Notes on Robust Control, H_∞, LMIs, BRL, Guaranteed Cost Control (with Delays) D-stability, Switched Systems

(includes Matlab code)

Leonidas D. Dritsas

Dipl. Electrical Eng., M.Sc, Ph.D

dritsas@aspete.gr

ldri@otenet.gr

http://users.otenet.gr/~ldri/index.htm

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Contents

1 References

Main References used for these notes

This presentation is based mainly on the following courses and textbooks: [1], [2], [3], [4] and [5]. Lecture notes are freely available at the web sites listed below.

1. Carsten Scherer: "LMIs in Controller Analysis and Synthesis". Lecture notes available at http://www.dcsc.tudelft.nl/~cscherer/lmi.html

2. Mauricio C. de Oliveira: Lecture notes available at http://www.dt.fee.unicamp.br/~mauricio/courses/ia360.html

3. Didier Henrion: "Course on LMI optimization with applications in control". Lecture notes available at http://www.laas.fr/~henrion/courses/lmi08/

4. Stephen Boyd, Laurent El Ghaoui, Eric Feron, & Venkataramanan Balakrishnan, "Linear Matrix Inequalities in System and Control Theory", SIAM Studies in Applied Mathematics, Philadelphia, PA, 1994. Book available at http://www.stanford.edu/~boyd/lmibook

Section 9 ("Guaranteed Cost Control (GCC) of Uncertain Discrete Time Systems with State and Input Delays"), is primarily based on [6] (L. Yu and F. Gao "Optimal guaranteed cost control of discrete-time, uncertain systems with both state and input delays", Journal of the Franklin Institute, vol. 338, 2001, p.101–110).

Section 12 is primarily based on [7] (X. Guan, Z. Lin and G. Duan "Robust guaranteed cost control for discrete-time, uncertain systems with delay", IEE Proc.-Control Theory Appl., vol. 146, November 1999, p.598–602).

2 Mathematical Background

2.1 SVD

The following are standard properties for the singular values of any matrices (X and Y of appropriate dimensions):

$$\sigma_{\max}(X+Y) \leq \sigma_{\max}(X) + \sigma_{\max}(Y) \quad (1)$$
$$\sigma_{\max}(XY) \leq \sigma_{\max}(X)\sigma_{\max}(Y) \quad (2)$$

(the "triangle inequality" and "Schwarz inequality" respectively).

Lemma 1 *For any two symmetric and positive–definite matrices X, Y the following relation can be proved*

$$\sigma_{\max}(X) < \sigma_{\min}(Y) \Rightarrow X < Y \, . \quad (3)$$

Sketch of Proof: It is known that every symmetric positive–definite (?? and "simple"??) matrix X can always be decomposed as $X = U\Lambda U^T$ with U being a unitary matrix ($UU^T = I$) and Λ a diagonal matrix with the eigenvalues as diagonal elements. Since X is also assumed symmetric we have $X^T = X = U\Lambda U^T$ and hence $XX^T = U\Lambda U^T U\Lambda U^T = U\Lambda^2 U^T$. Now...

...since $\sigma(X) \triangleq [\lambda(XX^T)]^{\frac{1}{2}}$ we conclude that for symmetric positive–definite matrices the set of its singular values coincides with the set of its strictly positive eigenvalues. On the other hand every positive–definite matrix X satisfies Raleigh's inequality (see (4)) $\lambda_{\min}(X)\,\|x\|^2 \leq x^T X x \leq \lambda_{\max}(X)\,\|x\|^2$. The quadratic $x^T(X-Y)x$ can now be bounded as follows: $x^T(X-Y)x = x^T(X)x - x^T(Y)x \leq \lambda_{\max}(X)\,\|x\|^2 - \lambda_{\min}(Y)\,\|x\|^2 = (\lambda_{\max}(X) - \lambda_{\min}(Y))\,\|x\|^2 = (\sigma_{\max}(X) - \sigma_{\min}(Y))\,\|x\|^2 < 0$, where the last inequality follows from the assumption $\sigma_{\max}(X) < \sigma_{\min}(Y)$ and the proved equality between the singular values and the (strictly positive) eigenvalues of SPD matrices. Hence, by definition, $X < Y$.

2.2 Properties of Real Symmetric (Hermitian) matrices

"Hermitian" matrices correspond to "Real Symmetric" when elements are real numbers.

- Let $A^H = \bar{A}^T$, $x^H = \bar{x}^T$, i.e. complex conjugate transpose. Matrix A is Hermitian if $A = A^H \Leftrightarrow x^H A x$ is real for all $x \in C^n$.
- Hermitian matrix D (i.e. $D = D^H$) is positive definite if $x^H D x > 0$ for all $x \neq 0$.
- For Hermitian D, its eigenvalues are real. Furthermore if D is real ("real symmetric") the eigenvectors are real as well.
- A Hermitian D is also "simple" (i.e. distinct eigenvalues, $\lambda_i \neq \lambda_j$) and eigenvectors corresponding to distinct eigenvalues are orthogonal ($x_j^H x_i = 0$).
- For Hermitian D, the eigenvector matrix can be written as a unitary matrix, that is $D = Q\Lambda Q^H$, $QQ^H = Q^H Q = I$ with Λ real and Q real if D real symmetric.
- For a Hermitian matrix D, we have D **positive definite,** ($D > 0$) if and only if ($\Leftrightarrow$) $\lambda_i > 0, \forall i$.

- Raleigh's inequalities: For any Symmetric (Hermitian) matrix A

$$\begin{aligned} \lambda_{\min}(A)I &\leq A \leq \lambda_{\max}(A)I \\ \sigma^2_{\min}(A)I &\leq A^T A \leq \sigma^2_{\max}(A) \\ \lambda_{\min}(A)x^T x &\leq x^T A x \leq \lambda_{\max}(A)x^T x \end{aligned}$$

- For any two Hermitian matrices M and N,

$$\lambda_{\min}(M+N) \geq \lambda_{\min}(M) + \lambda_{\min}(N)$$
$$\lambda_{\max}(M+N) \leq \lambda_{\max}(M) + \lambda_{\max}(N)$$

2.3 Properties of Positive Definite Matrices

- Addition of positive matrices: $A > 0$ and $B > 0 \Rightarrow A + B > 0$
- Block diagonal matrices: $A > 0$ and $B > 0 \Leftrightarrow \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} > 0$
- Invertibility: $A > 0 \Rightarrow A$ nonsingular.
- Convex cone property: $A,\ B \geq 0$ and $\lambda, \mu > 0 \Rightarrow \lambda A + \mu B \geq 0$.
  The set of positive semidefinite matrices is a convex cone.

Recall also that

$$\text{if } \ P = P^T > 0 \ \text{ then } \ P^{-1} \text{ always exists, and moreover } \ P^{-1} = (P^{-1})^T > 0 \tag{4}$$

2.4 Congruent Transformations

A Congruent Transformation preserves definiteness

- Suppose $X \in R^{n \times m}$. Then for any Symmetric (Hermitian) matrix A

$$A \geq 0 \Rightarrow X^T A X \geq 0$$

- Suppose $X \in R^{n \times m}$ and $Image\{X\} = R^n$. Then for any Symmetric (Hermitian) matrix A

$$A \geq 0 \Leftrightarrow X^T A X \geq 0$$

- Suppose $X \in R^{n \times m}$ and $Ker\{X\} = \{0\}$. Then for any Symmetric (Hermitian) matrix A

$$A > 0 \Rightarrow X^T A X > 0 \tag{5}$$

The inverse ($\Leftarrow$) is NOT true

2.5 Useful Matrix Inequalities

The (norm-2 induced vector-) norm $||\Delta||$ of a matrix is defined as

$$||\Delta||_2 \triangleq \sigma_{\max}(\Delta) = \sqrt{\lambda_{\max}(\Delta^T \Delta)},$$

and satisfies

$$||\Delta|| \leq \gamma \Leftrightarrow \sigma^2_{\max}(\Delta) = \lambda_{\max}(\Delta^T \Delta) \leq \gamma^2 \Leftrightarrow \Delta^T \Delta \leq \gamma^2 I$$

2.5.1 "First Fundamental Matrix Inequality"

The following inequality (and many variations of it) has been extensively used in the Robust and H_∞ Control literature (where Δ usually denotes unstructured uncertainty, and X, Y, Δ of compatible dimensions)

Lemma 2 *If $\Delta^T\Delta \le Q_\Delta$, then for any $\alpha > 0$,*

$$X^T\Delta Y + Y^T\Delta^T X \le \alpha X^T X + (\frac{1}{\alpha})Y^T\Delta^T\Delta Y \le \alpha X^T X + (\frac{1}{\alpha})Y^T Q_\Delta Y \tag{6}$$

Proof 3 *Assuming $\epsilon_1, \epsilon_2 > 0$, expand the right part of the (trivial) inequality* $0 \le [\epsilon_1 X - \frac{1}{\epsilon_2}\Delta Y]^T[\epsilon_1 X - \frac{1}{\epsilon_2}\Delta Y]$ *into* $0 \le [\epsilon_1 X - \frac{1}{\epsilon_2}\Delta Y]^T[\epsilon_1 X - \frac{1}{\epsilon_2}\Delta Y] = \epsilon_1^2 X^T X + \frac{1}{\epsilon_2^2}Y^T\Delta^T\Delta Y - \frac{\epsilon_1}{\epsilon_2}[X^T\Delta Y + Y^T\Delta^T X]$ $\Leftrightarrow \frac{\epsilon_1}{\epsilon_2}[X^T\Delta Y + Y^T\Delta^T X] \le \epsilon_1^2 X^T X + \frac{1}{\epsilon_2^2}Y^T\Delta^T\Delta Y]$. *The proof is completed by multiplying both sides by $\frac{\epsilon_2}{\epsilon_1} > 0$ and then by setting $\alpha = \epsilon_1\epsilon_2 > 0$, while using $\Delta^T\Delta \le Q_\Delta$.*

If it is further assumed that Δ is norm bounded as

$$||\Delta||_2 = \sigma_{\max}(\Delta) \le \gamma \Leftrightarrow \Delta^T\Delta \le \gamma^2 I$$

the previous inequality becomes

$$X^T\Delta Y + Y^T\Delta^T X \le \alpha X^T X + (\frac{\gamma^2}{\alpha})Y^T Y, \ \ ||\Delta|| \le \gamma \tag{7}$$

Remark 4 *For the admissible choices $\alpha = \gamma$ or $\alpha = \gamma^2$, inequality (7) becomes respectively*

- $X^T\Delta Y + Y^T\Delta^T X \le \gamma X^T X + \gamma Y^T Y,\ ||\Delta||_2 \le \gamma$
- $X^T\Delta Y + Y^T\Delta^T X \le \gamma^2 X^T X + Y^T Y,\ ||\Delta||_2 \le \gamma$

2.5.2 Variations of the "First Fundamental Matrix Inequality"

Many variations of the "Fundamental inequality" appear in the literature of Uncertain (and Time Delayed) Systems e.g.

1.
$$0 \le [Z^{1/2}X - \frac{1}{\alpha}Z^{-1/2}Y]^T[Z^{1/2}X - \frac{1}{\alpha}Z^{-1/2}Y] \Leftrightarrow$$
$$X^T Y + Y^T X \le \alpha X^T Z X + (\frac{1}{\alpha})Y^T Z^{-1} Y \quad \text{with} \ \ \alpha,\ Z > 0$$

   (which is proved by expanding $0 \le [Z^{1/2}X - \frac{1}{\alpha}Z^{-1/2}Y]^T[Z^{1/2}X - \frac{1}{\alpha}Z^{-1/2}Y] \Leftrightarrow$ $X^T Y + Y^T X \le \alpha X^T Z X + (\frac{1}{\alpha})Y^T Z^{-1} Y \quad \text{with} \ \ \alpha,\ Z > 0.$)

2.
$$\begin{aligned} & X^T Z Y + Y^T Z^T X = \\ & \alpha X^T X + \frac{1}{\alpha}Y^T Z^T Z Y - \alpha[X - \frac{1}{\alpha}ZY]^T[X - \frac{1}{\alpha}ZY] \\ & \le \alpha X^T X + \frac{1}{\alpha}Y^T Z^T Z Y \quad \text{with} \ \ \alpha,\ Z > 0 \end{aligned}$$

which can be easily proved by noticing that

$$0 \leq \alpha \left[X - \frac{1}{\alpha}ZY\right]^T \left[X - \frac{1}{\alpha}ZY\right] = \alpha \left[X^T X + \frac{1}{\alpha^2} Y^T Z^T ZY - \frac{1}{\alpha} X^T ZY - \frac{1}{\alpha} Y^T Z^T X\right]$$
$$= \alpha X^T X + \frac{1}{\alpha} Y^T Z^T ZY - X^T ZY - Y^T Z^T X$$
$$\Leftrightarrow X^T ZY + Y^T Z^T X = \alpha X^T X + \frac{1}{\alpha} Y^T Z^T ZY - \alpha \left[X - \frac{1}{\alpha}ZY\right]^T \left[X - \frac{1}{\alpha}ZY\right]$$
$$\alpha,\ Z > 0$$

3. **more variations to be written...**

4. ...

2.5.3 "Second Fundamental Matrix Inequality (Lemma)"

(Wang & Xie 1992, Xie 1996, Petersen 1987) see

Lemma 5 *Given matrices G, M, N of compatible dimensions with G symmetric, the inequality*

$$G + M\Delta N + N^T \Delta^T M^T < 0$$

*holds for all Δ satisfying $\Delta^T \Delta \leq R$ **if and only if (⇔)** there exists a constant $\epsilon > 0$ such that*

$$G + \epsilon M M^T + \frac{1}{\epsilon} N^T R N < 0$$

Remark 6 *Using Schur's complements the last inequality can be equivalently written as (set $R = I$ for simplicity)*

$$G + \epsilon^2 M M^T + \frac{1}{\epsilon^2} N^T N < 0 \quad \Leftrightarrow \quad \begin{bmatrix} G & \begin{pmatrix} \frac{1}{\epsilon} N^T & \epsilon M \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\epsilon} N \\ \epsilon M^T \end{pmatrix} & -I_{2n} \end{bmatrix} < 0$$
$$\Leftrightarrow \quad \begin{bmatrix} G & \begin{pmatrix} \epsilon M & \frac{1}{\epsilon} N^T \end{pmatrix} \\ \begin{pmatrix} \epsilon M^T \\ \frac{1}{\epsilon} N \end{pmatrix} & -I_{2n} \end{bmatrix} < 0$$

Variation with $\sigma_{\max}(\Delta) \leq \delta \Leftrightarrow \Delta^T \Delta \leq \delta^2 I$ also useful...

Lemma 7 *Given matrices G, M, N of compatible dimensions with G symmetric, then*

$$G + M\Delta N + N^T \Delta^T M^T < 0$$

*holds for all Δ satisfying $\sigma_{\max}(\Delta) \leq \delta$ **if and only if** there exists a constant $\epsilon > 0$ such that*

$$G + \epsilon M M^T + \frac{\delta^2}{\epsilon} N^T N < 0$$

2.5.4 "Third Fundamental Matrix Inequality" and the Matrix Inversion Lemma

The inequality presented in Lemma 9 below (and variations) has also been extensively used in the Robust Control literature Its proof needs the "Matrix Inversion Lemma" which states that

Lemma 8

$$[A_1 + A_2A_3A_4]^{-1} = A_1^{-1} - A_1^{-1}A_2[A_4A_1^{-1}A_2 + A_3^{-1}]^{-1}A_4A_1^{-1} \tag{8}$$

Lemma 9 *Let A, M, N, Δ be real matrices of appropriate dimensions with $\|\Delta\|_2 < 1$. Then for $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - M^TPM > 0$,*

$$(A + M\Delta N)^TP(A + M\Delta N) \leq A^TPA + A^TPM(\varepsilon I - M^TPM)^{-1}M^TPA + \varepsilon N^TN \tag{9}$$

Sketch of Proof of inequality (9) in Lemma 9.

Proof 10 *Start by forming the "square" $0 \leq Y^TY$ where*
$Y \triangleq [\varepsilon I - M^TPM]^{-\frac{1}{2}}M^TPA - [\varepsilon I - M^TPM]^{\frac{1}{2}}\Delta N$ *and hence*
$Y^T = A^TPM[\varepsilon I - M^TPM]^{-\frac{1}{2}} - N^T\Delta^T[\varepsilon I - M^TPM]^{\frac{1}{2}}$.

Forming the square Y^TY we get:
$0 \leq Y^TY = A^TPM[\varepsilon I - M^TPM]^{-1}M^TPA - A^TPM\Delta N - N^T\Delta^TM^TPA + N^T\Delta^T[\varepsilon I - M^TPM]\Delta N \Leftrightarrow$

$A^TPM\Delta N + N^T\Delta^TM^TPA \leq A^TPM[\varepsilon I - M^TPM]^{-1}M^TPA +$
$N^T\Delta^T[\varepsilon I - M^TPM]\Delta N$.

Adding now the "missing terms" $A^TPA + (N^T\Delta^TM^T)P(M\Delta N)$ to both sides of the last inequality, the LHS becomes the "complete square" $(A + M\Delta N)^TP(A + M\Delta N)$ and hence

$(A + M\Delta N)^TP(A + M\Delta N) \leq A^TPA + A^TPM[\varepsilon I - M^TPM]^{-1}M^TPA + N^T\Delta^T[\varepsilon I - M^TPM]\Delta N +$
$(N^T\Delta^TM^T)P(M\Delta N)$.

The last two terms are bounded as
$N^T\Delta^T[\varepsilon I - M^TPM]\Delta N + (N^T\Delta^TM^T)P(M\Delta N) = N^T\Delta^T[\varepsilon I - M^TPM + M^TPM]\Delta N =$
$\varepsilon N^T\Delta^T\Delta N \leq \varepsilon N^TN$ *and the (sketch of) proof is complete.*

Applying the "Matrix Inversion Lemma" (8) to the expression $(P^{-1} - \varepsilon^{-1} - MM^T)^{-1}$ in (9) of Lemma 9 we get a useful alternative of (9) since $(P^{-1} - \varepsilon^{-1} - MM^T)^{-1} = P - PM[M^TPM - \varepsilon I]^{-1}M^TP$ and hence $A^T[P^{-1} - \varepsilon^{-1}MM^T]^{-1}A = A^TPA + A^TPM(\varepsilon I - M^TPM)^{-1}M^TPA$. Inequality (9) can then be equivalently expressed as

Lemma 11 *Let A, M, N, Δ be real matrices of appropriate dimensions with $\|\Delta\|_2 < 1$. Then for $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - M^TPM > 0$,*

$$(A + M\Delta N)^TP(A + M\Delta N) \leq A^T[P^{-1} - \varepsilon^{-1}MM^T]^{-1}A + \varepsilon N^TN \tag{10}$$

The above inequality is also used with P in lieu of P^{-1} and ε^{-1} in lieu of ε i.e.

Lemma 12 *Let A, M, N, Δ be real matrices of appropriate dimensions with $\|\Delta\|_2 < 1$. Then for $P > 0$ and scalar $\varepsilon > 0$ satisfying $P - \varepsilon MM^T > 0$*

$$(A + M\Delta N)^T P^{-1}(A + M\Delta N) \leq A^T[P - \varepsilon MM^T]^{-1}A + \frac{1}{\varepsilon}N^T N \qquad (11)$$

See also Lemma (12).

Another useful inequality is derived by expanding $(A + M\Delta N)^T P(A + M\Delta N)$ in (9) and then cancelling the term $A^T PA$ from both sides. Indeed since $(A + M\Delta N)^T P(A + M\Delta N) = A^T PA + A^T PM\Delta N + N^T\Delta^T M^T PA + (N^T\Delta^T M^T)P(M\Delta N)$ and (9) immediately yields

Lemma 13 *Let A, M, N, Δ be real matrices of appropriate dimensions with $\|\Delta\|_2 < 1$. Then for $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - M^T PM > 0$,*

$$A^T PM\Delta N + N^T\Delta^T M^T PA + (N^T\Delta^T M^T)P(M\Delta N) \leq A^T PM(\varepsilon I - M^T PM)^{-1}M^T PA + \varepsilon N^T N \qquad (12)$$

2.6 Schur Complement and LMIs

Lemma 14 *The following statements are equivalent ("$\Leftrightarrow$")*

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} < 0 \Leftrightarrow \begin{cases} Z < 0 \ \ and \ \ X - YZ^{-1}Y^T < 0 \\ X < 0 \ \ and \ \ Z - Y^T X^{-1} Y < 0 \\ \begin{bmatrix} Z & Y^T \\ Y & X \end{bmatrix} < 0, \end{cases} \qquad (13)$$

Schur's Lemma (13) also valid with ">" inequality sign.

Remark 15

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} < 0 \Rightarrow X < 0 \ \ and \ \ Z < 0$$

The following Lemmas are Direct Application of Schur's Lemma (13)

Lemma 16 *($\epsilon > 0$)*

$$\begin{bmatrix} A & B & D^T \\ B^T & C & E^T \\ D & E & -\epsilon I \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} + \frac{1}{\epsilon}\begin{bmatrix} D^T \\ E^T \end{bmatrix}\begin{bmatrix} D & E \end{bmatrix} < 0$$

$$\Leftrightarrow \quad \begin{bmatrix} -\epsilon I & D & E \\ D^T & A & B \\ E^T & B^T & C \end{bmatrix} < 0$$

*(**Sketch of Proof:** Conceive the RHS as $X - YZ^{-1}Y^T < 0$ with $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, $Y = \begin{bmatrix} D^T \\ E^T \end{bmatrix}$, $Z = -\epsilon I < 0$ and apply Schur's Lemma)*

Lemma 17 *(Generalization of Lemma 16 with $P > 0$)*

$$\begin{bmatrix} A & B & D^T \\ B^T & C & E^T \\ D & E & -P \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} + \begin{bmatrix} D^T \\ E^T \end{bmatrix} \left(P^{-1}\right) \begin{bmatrix} D & E \end{bmatrix} < 0$$

$$\Leftrightarrow \quad \begin{bmatrix} -P & D & E \\ D^T & A & B \\ E^T & B^T & C \end{bmatrix} < 0$$

3 Lyapunov Stability Analysis and Static State Feedback (SSF) Synthesis via LMI

3.1 Continuous Time (CT) Lyapunov inequality expressed as LMI

Recall that the Continuous–Time (CT) LTI system $\dot{x}(t) = Ax(t)$, is exponentially stable **iff** there exists $P = P^T > 0$ such that

$$A^T P + PA < 0, \quad P = P^T > 0$$

Taking into account that

- The Lyapunov inequalities (1) $A^T P + PA < 0$ and (2) $P > 0$ are already in LMI format (depend affinely on P)
- multiple LMIs can be combined into a single LMI

$$\begin{bmatrix} A^T P + PA & 0 \\ 0 & -P \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -A^T P - PA & 0 \\ 0 & P \end{bmatrix} > 0 \tag{14}$$

Conclusion: The problem of stability verification for $\dot{x}(t) = Ax(t)$ is transformed into a strict feasibility test for the LMI (14), whose solution P (if exists) defines the Quadratic Lyapunov Function $V(x) = x^T Px$, which certifies asymptotic stability.

3.2 SSF Synthesis via LMI for CT–LTI systems without uncertainty

For the **CT LTI system** with Static ("memoryless") State Feedback

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t),$$

the closed–loop system is stable **iff** there exists $P = P^T > 0$ such that

$$(A + BK)^T P + P(A + BK) < 0. \tag{15}$$

The "congruence + change of variables" trick (20 years old !!!)

Though (15) is not an LMI in P, K it can be transformed into an LMI using the new variables $S = P^{-1} > 0$, $W = KS = KP^{-1}$. Recall that (i) if $P = P^T > 0$ then $S = P^{-1}$ always exists and is also symmetric positive definite ($S = S^T > 0$) and that (ii) a congruent transformation preserves definiteness.

Pre-, Post- multiplying (15) by S and using $W = KP^{-1} = KS$, $W^T = P^{-1}K^T = SK^T$,

$$\begin{aligned} S\left(A^T P + K^T B^T P + PA + PBK\right)S &< 0 \Leftrightarrow \\ SA^T PS + SK^T B^T PS + SPAS + SPBKS &< 0 \Leftrightarrow \\ SA^T + W^T B^T + AS + BW &< 0 \end{aligned}$$

which is an LMI in S, W !!!

The algorithm for Continuous–Time State Feedback design via LMI is thus:

Solve the LMI feasibility problem

$$\begin{aligned} -S &< 0 \\ SA^T + W^T B^T + AS + BW &< 0 \end{aligned} \tag{16}$$

for S, W and (if solution exists) get the static state feedback gain as $K = WS^{-1}$. Moreover the quadratic Lyapunov function $x^T Px = x^T S^{-1} x$ proves closed loop stability.

3.3 SSF Synthesis via LMI for CT–LTI polytopic uncertain systems

Robust control deals with the problem of inexact or (intentionally) simplified system models and the design objective is that the robust controller will perform "satisfactorily" in terms of stability and performance when applied on the actual (uncertain) system

Problem Statement: Using the "standard Robust Control Paradigm" (Figure 1) the objective is to find an optimal control strategy $u(t) = Kx(t)$ or $u(t) = Ky(t)$ such that the closed–loop system enjoys good robustness properties. The uncertainties are (possibly) measurable in real–time, and can (in this case) be used for feedback, i.e "gain–scheduled"

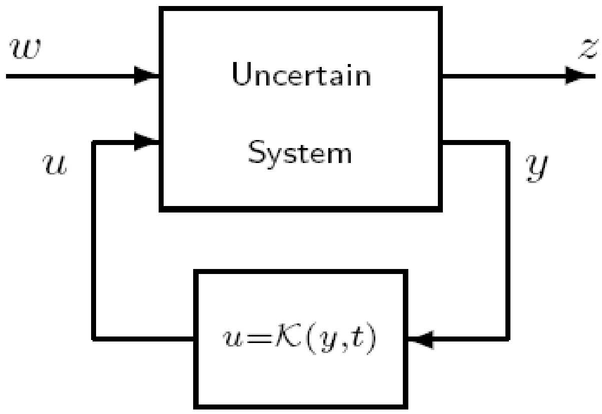


Figure 1: Robust Control Paradigm

The above methodology , expressed via the (16) LMI, can be generalized for the robust stabilization via state feedback of a polytopic uncertain system as following:

Consider the polytopic system with static state feedback

$$\dot{x}(t) = A(t)x(t) + Bu(t), \;\; u(t) = Kx(t), \;\; A(t) \in \mathbf{co}(A_1, \dots, A_L)$$

The closed–loop system is (quadratically) stable **iff**

$$P = P^T > 0, \;\; (A_i + BK)^T P + P(A_i + BK) < 0, \;\; i = 1, ..., L$$

and the corresponding LMI methodology now becomes

$$\begin{aligned} -S &< 0 \\ SA_i^T + W^T B^T + A_i S + BW &< 0, \;\; i = 1, ..., L \end{aligned} \tag{17}$$

Output feedback strategy $u(t) = Ky(t)$ on the other hand is in general "much harder" than state feedback. This difficulty can be illustrated by presenting the case of constant output feedback for polytopic systems.

$$\dot{x} = A(t)x(t) + Bu(t), \quad y(t) = Cx(t), \quad u(t) = Ky(t), \; A(t) \in \mathbf{Co}(A_1, ..., A_L)$$

where the closed–loop system is stable if

$$P = P^T > 0, \;\; (A_i + BKC)^T P + P(A_i + BKC) < 0, \;\; i = 1, ..., L \tag{18}$$

Output feedback strategy $u(t) = Ky(t)$ gives LMIs only in a handful of cases since no "Convexification" is possible in general.

3.4 SSF Synthesis via LMI for CT–LTI systems with norm–bounded uncertainty

Open–loop CT system dynamics with norm–bounded uncertainties in both the system and the input matrices, i.e.

$$\dot{x} = (A + \Delta A)x(t) + (B + \Delta B)u(t), \quad x \in \Re^n, \; u \in \Re^m \tag{19}$$

with

$$[\Delta A \quad \Delta B] = D_a F \, [E_a \;\; E_b] \tag{20}$$

D_a, E_a, E_b known and constant, and the unknown (time-varying) matrix $F(t)$ satisfying $F^T F \leq I$. The closed–loop dynamics with $u(t) = Kx(t)$ are

$$\dot{x} = [A + BK + D_a F(E_a + E_b K)] \, x_k \stackrel{\triangle}{=} A_C x(t) \tag{21}$$

with the obvious definition for the uncertain closed-loop matrix A_C .

Defining the "quadratic candidate Lyapunov function function $V = x(t)^T P x(t)$, with $P > 0$ being a SPDef matrix of appropriate dimensions $0 < P^T = P \in \Re^{n \times n}$, the "wish" for $\dot{V} < 0$ along the trajectories of the closed-loop system can be equivalently expressed as $A_C^T P + P A_C < 0$ or

$$(A + BK)^T P + ((E_a + E_b K)^T F^T D_a^T)P + P(A + BK) + PD_a F(E_a + E_b K) < 0$$

Introducing $S = P^{-1}$, $W = KP^{-1} = KS$, and Pre-, Post- multiplying the last inequality by S, can write equivalently

$$\begin{aligned} S(A + BK)^T PS + SP(A + BK)S + S[((E_a + E_b K)^T F^T D_a^T)P]S + SPD_a F(E_a + E_b K)S < 0 \Leftrightarrow \\ S(A + BK)^T + (A + BK)S + S^T [D_a F(E_a + E_b K)]^T + D_a F(E_a + E_b K)S < 0 \Leftrightarrow \\ (AS + BW) + (SA^T + W^T B^T) + D_a F(E_a S + E_b W) + (E_a S + E_b W)^T F^T D^T < 0 \end{aligned}$$

The last matrix inequality is clearly of the form $G + M\Delta N + N^T \Delta^T M^T < 0$, with $G^T = G \stackrel{\triangle}{=} (AS + BW) + (SA^T + W^T B^T)$ and hence...,

Lemma 5 can be used to transform it into the following equivalent "$G + \epsilon M M^T + \frac{1}{\epsilon} N^T R N$" inequality (valid $\forall$ admissible F and $\epsilon > 0$) i.e.

$$\begin{aligned} G + \epsilon D_a D_a^T + \frac{1}{\epsilon}(E_a S + E_b W)^T (E_a S + E_b W) < 0 \Leftrightarrow \\ (G + \epsilon D_a D_a^T) - (E_a S + E_b W)^T (-\epsilon I_n)^{-1} (E_a S + E_b W) < 0 \Leftrightarrow \end{aligned}$$

$$\begin{bmatrix} (AS + BW) + (SA^T + W^T B^T) + \epsilon D_a D_a^T & (E_a S + E_b W)^T \\ (E_a S + E_b W) & -\epsilon I_n \end{bmatrix} < 0 \tag{22}$$

If this last LMI has a feasible solution, in terms of the variables $\{\epsilon$, W, S $\}$, then the state feedback control law $u(t) = WS^{-1}x(t) = Kx(t)$ is a robustly stabilizing control law.

3.5 Discrete Time (DT) Lyapunov inequality expressed as LMI

Recall that the **Discrete–Time (DT) LTI system** $x_{k+1} = Ax_k$, is exponentially stable **iff** there exists $P = P^T > 0$ such that $A^T PA - P < 0$. Manipulating the Lyapunov inequalities $A^T PA - P < 0, \;\; P = P^T > 0$ as follows:

$$\begin{aligned} A^T PA - P < 0, \;\; P = P^T > 0 \Leftrightarrow \\ -P + A^T PA < 0, \;\; -P < 0 \Leftrightarrow \\ -P - A^T(-P^{-1})^{-1}A < 0, \;\; -P < 0 \end{aligned} \tag{23}$$

and using Schur's Lemma 14 (version $Z - Y^T X^{-1} Y < 0, \;\; X < 0$ with $Z \to -P, \;\; Y \to A, \;\; X \to -P^{-1} \;\; \Leftrightarrow \;\; X^{-1} \to -P < 0$) can express the Lyapunov inequalities $A^T PA - P < 0, \;\; P = P^T > 0$ as

$$\begin{bmatrix} -P^{-1} & A \\ A^T & -P \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -P & A^T \\ A & -P^{-1} \end{bmatrix} < 0 \tag{24}$$

Remark 18 *Working with the ">" version of DT Lyapunov we equivalently have* $A^T PA - P < 0 \Leftrightarrow P - A^T PA > 0 \Leftrightarrow$ *(Schur's Lemma)*

$$\begin{bmatrix} P^{-1} & A \\ A^T & P \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} P & A^T \\ A & P^{-1} \end{bmatrix} > 0 \tag{25}$$

Note that (24), (25) are NOT LMIs ! hence **an alternative manipulation** is needed...

DT Lyapunov LMI 1: A **first alternative expression** can be derived via direct manipulation of (23) while recalling that $P = P^T > 0$

$$\begin{aligned} A^T PA - P < 0, \;\; P = P^T > 0 \Leftrightarrow \\ P - A^T(PP^{-1})PA > 0, \;\; P > 0 \Leftrightarrow \\ \underbrace{P - (PA)^T(P^{-1})(PA) > 0, \;\; P > 0 \Leftrightarrow}_{\substack{\text{use version } Z - Y^T X^{-1} Y > 0, \; X > 0 \\ \text{of Schur's Lemma}}} \end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} P & PA \\ A^T P & P \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} > 0 \tag{26}$$

Now (26) is a "feasibility" LMI on matrix variable P.

DT Lyapunov LMI 2: A **second alternative expression** can be derived by pre– and post–multiplying (23) by $S \triangleq P^{-1}$ (a congruent transformation with $S \triangleq P^{-1}$) while recalling that

- if $P = P^T > 0$ then $S = P^{-1}$ always exists and is also SPD (i.e. $S = S^T > 0$)
- a congruent transformation preserves definiteness

Thus

$$-P < 0 \Leftrightarrow -P^{-1}PP^{-1} < 0 \Leftrightarrow -P^{-1} = -S < 0$$

and

$$\begin{aligned} -P + A^T PA < 0 \quad &\Leftrightarrow \quad P^{-1}\left[-P + A^T PA\right]P^{-1} < 0 \\ &\Leftrightarrow \quad -P^{-1} - (AP^{-1})^T(-P^{-1})^{-1}(AP^{-1}) < 0 \\ &\Leftrightarrow \quad \underbrace{-S - (AS)^T(-S)^{-1}(AS) < 0}_{Z - Y^T X^{-1} Y < 0, \ \ X < 0} \end{aligned}$$

(use Schur's Lemma 14 with $Z \to -P^{-1} = -S, \quad Y \to AP^{-1} = AS, \quad X \to -P^{-1} = -S < 0$),

$$\Leftrightarrow \begin{bmatrix} -S & AS \\ SA^T & -S \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -S & SA^T \\ AS & -S \end{bmatrix} < 0, \ \ S \triangleq P^{-1} \tag{27}$$

Now (27) is a "feasibility" LMI on matrix variable S.

Remark 19 *The same result (27) could have been directly derived from (24) via the following congruent transformation:*

$$\begin{bmatrix} -P^{-1} & A \\ A^T & -P \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} I & 0 \\ 0 & P^{-1} \end{bmatrix}\begin{bmatrix} -P^{-1} & A \\ A^T & -P \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & P^{-1} \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} -P^{-1} & AP^{-1} \\ P^{-1}A^T & -P^{-1} \end{bmatrix} < 0 \ \ i.e. \ \begin{bmatrix} -S & AS \\ SA^T & -S \end{bmatrix} < 0$$

Conclusion: The problem of stability verification for the Discrete–Time LTI system $x_{k+1} = Ax_k$ is transformed into a strict LMI feasibility test for the LMIs (26) or (27).

These LMIs are two alternative & equivalent "tests" for the (Quadratic) Stability of the examined Discrete–Time LTI system.

The solution(s) $P = S^{-1}$ or S respectively, if exist, define the Quadratic Lyapunov Function $V(x_k) = x_k^T P x_k$, which certifies asymptotic stability.

3.6 SSF Synthesis via LMI for DT–LTI systems without uncertainty

Fact: For the **DT LTI system** with static State Feedback

$$x_{k+1} = Ax_k + Bu_k, \quad u_k = Kx_k,$$

the closed–loop system is stable **iff** there exists $P = P^T > 0$ such that

$$(A + BK)^T P(A + BK) - P < 0. \tag{28}$$

which is not an LMI in P, K, but can be transformed into one by using the new variables $S = P^{-1} > 0, \ \ W = KP^{-1} = KS$ and... ...the "change of variables" trick again !!!: Pre-, Post- multiplying (28) by the symmetric positive definite matrix $S = P^{-1}$ and using the new variables $S = P^{-1} > 0, \ \ W = KP^{-1} = KS$, while recalling that "a congruent transformation preserves definiteness", Lyapunov inequality (28) transforms into

$$\begin{aligned}
S\left((A^T + K^T B^T)S^{-1}(A + BK) - S^{-1}\right)S &< 0 \Leftrightarrow \\
(SA^T + SK^T B^T)S^{-1}(A + BK)S - SS^{-1}S &< 0 \Leftrightarrow \\
(SA^T + W^T B^T)S^{-1}(AS + BW) - S &< 0 \Leftrightarrow \\
-S - (SA^T + W^T B^T)(-S)^{-1}(AS + BW) &< 0
\end{aligned}$$

The last matrix inequality can be expressed as an LMI feasibility problem in terms of the matrix variables $S,\ W$ by invoking Schur's Lemma 14 (use the "$Z - Y^T X^{-1} Y < 0,\ \ X < 0$" version with $Z \to -P^{-1} = -S < 0,\ \ Y \to (AS + BW),\ \ X \to -S$)

$$-S - (SA^T + W^T B^T)(-S)^{-1}(AS + BW) < 0 \Leftrightarrow$$

$$\begin{bmatrix} -S & AS + BW \\ SA^T + W^T B^T & -S \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -S & SA^T + W^T B^T \\ AS + BW & -S \end{bmatrix} < 0 \tag{29}$$

The algorithm for Discrete–Time State Feedback design via LMI is thus:

Solve the LMI feasibility problem (29) for $S,\ W$ and then get the static state feedback gain as $K = WS^{-1}$.

3.7 SSF Synthesis via LMI for DT–LTI systems with polytopic or norm–bounded uncertainty

The procedure presented in previous section can be generalized for discrete time polytopic systems as done for the CT case (see (17)).

The norm–bounded uncertainty case is covered in another section (see 10.4 further down).

4 Bounded Real Lemma for Continuous Time systems (CT-BRL)

4.1 BRL as a Robust Quadratic Stability Criterion for CT-LTI Uncertain Systems

The BRL presentation starts by investigating the Robust Quadratic Stability of the autonomous continuous–time LTI uncertain (CTLTI) System

$$\dot{x}(t) = A(\Delta)x(t) = (A_0 + \Delta A)x(t) = (A_0 + M_A\Delta_A N_A)x(t) \tag{30}$$

Interpretation of the structure of the uncertain system matrix $A(\Delta) = A_0 + M_A\Delta_A N_A$:

- A_0 is the nominal (known and time–invariant) system matrix,
- M_A, N_A are known constant matrices capturing the uncertainty **structure**
- Δ_A being an uncertain matrix capturing the uncertainty **magnitude** via the norm bound relation $\sigma_{max}(\Delta_A) \leq \delta_A \triangleq \frac{1}{\gamma_A}$, i.e.

Recall that $\sigma_{max}(\Delta) = ||\Delta|| \leq \gamma \Leftrightarrow \lambda_{\max}(\Delta^T\Delta) \leq \gamma^2 \Leftrightarrow \Delta^T\Delta \leq \gamma^2 I$

Remark 20 *It is intuitively expected that the nominal matrix A_0 has to be (Hurwitz) stable...otherwise...for the admissible value $\Delta = 0$... See remark (24) below*

Remark 21 *In most of the presentation below,* ***for notation convenience*** *only, we* ***temporarily*** *omit the "A" subscripts from M_A, Δ_A, N_A writing simply M, Δ, N. Similarly we temporarily omit the "0" subscript from the nominal matrix A_0 writing simply "A".*

The problem is thus formulated as: Investigate the conditions for robust quadratic stability of the norm bounded uncertain system

$$\dot{x}(t) = A(\Delta)x(t) = [A + M\Delta N]x(t), \quad \text{with} \quad \sigma_{max}(\Delta) \leq \delta \triangleq \frac{1}{\gamma} \tag{31}$$

Definition: The CTLTI uncertain system (31) is said to be robustly stable if (31) is asymptotically stable for all $A(\Delta)$.

Alternative Definition: Let Ω be the set of stable ("Hurwitz") matrices i.e.

$$\Omega \triangleq X \in \Re^{n\times n} : maxRe\{\lambda(X)\} < 0$$

Robust stability is then (re)defined as follows:"If $A \subseteq \Omega$ then the CTLTI uncertain system (31) is robustly stable".

Problem: The set of all stable matrices Ω is not a convex set (it is a non convex cone).

Remark 22 *Note that (30),(31) can be written as the feedback interconnection of the system*

$$\dot{x}(t) = Ax(t) + Mw(t), \quad z(t) = Nx(t), \tag{32}$$

with the uncertain element $w(t) = \Delta z$ (see figure 2 below). This "feedback representation" will later clarify ***the relation between BRL and the Small Gain Theorem****.*

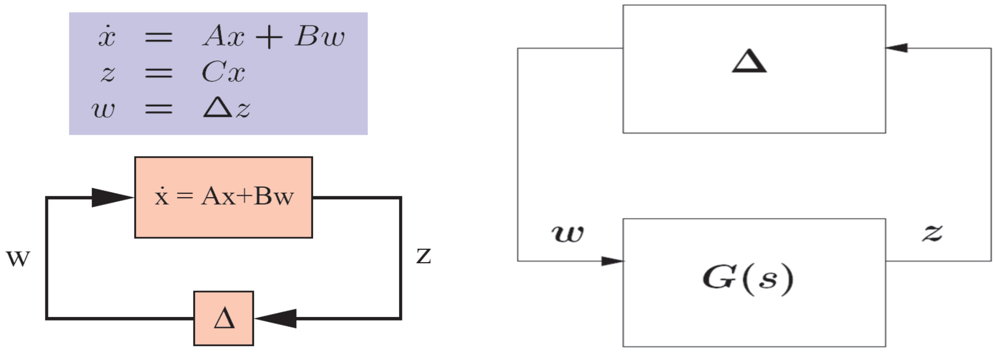


Figure 2: Relating BRL to SmallGain Theorem

Taking the time derivative of $V(x) = x^T Px$, $P = P^T > 0$ along the trajectories of (31),(32)

$$\begin{aligned} \dot{V}(x(t)) &= 2x^T P[A + M\Delta N]x \\ &= x^T[A^T P + PA + PM\Delta N + N^T \Delta^T M^T P]x \end{aligned}$$

and using the "fundamental inequality" (7) along with the assumption $||\Delta|| \leq \delta$, the term $PM\Delta N + N^T \Delta^T M^T P$ is bounded as

$$\underbrace{(PM)\Delta N + N^T\Delta^T(M^T P) \leq \alpha PMM^T P + (\frac{\delta^2}{\alpha})N^T N}_{X^T\Delta Y \; + \; Y^T\Delta^T X \leq \alpha X^T X + (\frac{\delta^2}{\alpha})Y^T Y}$$

Using the admissible value $\alpha = \delta > 0$ for the positive "tuning (scalar) variable" α, the previous inequality (bound) becomes

$$(PM)\Delta N + N^T\Delta^T(M^T P) \leq \delta PMM^T P + \delta N^T N$$

and hence $\dot{V}(x(t)) < x^T[A^T P + PA + \delta PMM^T P + \delta N^T N]x$

4.1.1 First Version of BRL (no feed–through term)

It is now clear clear that a sufficient condition for the (Robust Quadratic) Asymptotic Stability for (31) is

$$A^T P + PA + \delta PMM^T P + \delta N^T N < 0, \quad P > 0 \tag{33}$$

Writing the last inequality as $A^T P + PA + \delta N^T N - PM(-\frac{1}{\delta}I)^{-1}(PM)^T < 0$ and using Schur's Lemma (with "Z" $\rightarrow -\frac{1}{\delta}I < 0$), the conditions in (33) are expressed equivalently ($\Leftrightarrow$) as an LMI

$$\begin{bmatrix} A^T P + PA + \delta N^T N & PM \\ M^T P & (-\frac{1}{\delta}I) \end{bmatrix} < 0, \quad P > 0 \tag{34}$$

LMI (34) is the first version of BRL in its "2×2" formulation, and its Feasibility guarantees Robust Quadratic Stability for the uncertain system (31).

Writing inequality (34) as

$$\underbrace{\begin{bmatrix} PA + A^T P & PM \\ M^T P & -\frac{1}{\delta} I \end{bmatrix} + \delta \begin{bmatrix} N^T \\ 0 \end{bmatrix} \begin{bmatrix} N & 0 \end{bmatrix}}_{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} D^T \\ E^T \end{bmatrix} \begin{bmatrix} D & E \end{bmatrix}} < 0$$

and using Lemma (16), we equivalently (⇔) get

$$\underbrace{\begin{bmatrix} PA + A^T P & PM & N^T \\ M^T P & -\frac{1}{\delta} I & 0 \\ N & 0 & -\frac{1}{\delta} I \end{bmatrix}}_{\begin{bmatrix} A & B & D^T \\ B^T & C & E^T \\ D & E & -\epsilon I \end{bmatrix}} < 0 \tag{35}$$

LMI (35), along with the (LMI) constraint $P > 0$, is the first version of BRL in its "3×3" formulation and is of course equivalent to LMI (34).

Remark 23 *Recall that*

- *"Robust Quadratic Stability" = "simultaneous stability"="a single Lyapunov matrix P proves stability for the whole uncertainty range"*
- *Recall: Quadratic stability ⇒ robust stability ⇒ vertex stability.* ***The converse is NOT true***

Remark 24 *Recalling the remark 15 on matrix definiteness, it is clear that the (1,1) element of the LMI (35) "demands" that $PA + A^T P < 0$ i.e. BRL expressed via LMI (35) demands that the nominal matrix A is "Hurwitz" stable !!!...See also the intuitive remark 20.*

4.1.2 First Version of BRL with feed–through term

Generalize the system dynamics in (32) by adding a "feed–through" term

$$\dot{x}(t) = Ax(t) + Mw(t), \quad z(t) = Nx(t) + Dw(t) \tag{36}$$

with the uncertain element $w(t) = \Delta z$. Provided that $(I - \Delta D)^{-1}$ exists, can write

$$w = \Delta\left(Nx(t) + Dw(t)\right) \Leftrightarrow w = (I - \Delta D)^{-1} \Delta N x \tag{37}$$

Substitution of $w = (I - \Delta D)^{-1} \Delta N x$ into (36) provides an **"LFT" representation of the uncertainty** (Linear Fractional Transformation)

$$\dot{x}(t) = [A + M(I - \Delta D)^{-1} \Delta N] x(t) \tag{38}$$

The requirement for non–singularity of $(I - \Delta D)$ is **a well posedeness requirement** which is equivalent to

$$(I - \Delta D) \text{ nonsingular } \forall \Delta \;\Leftrightarrow\; I > \delta^2 D^T D \tag{39}$$

Using the same math as before, the ("complete") version of BRL with feed–through term becomes

$$A^TP + PA + \delta N^TN + (PM + \delta N^TD)[\tfrac{1}{\delta}I - \delta D^TD]^{-1}(M^TP + \delta D^TN) < 0$$

$$\Updownarrow$$

$$P > 0, \quad \begin{bmatrix} A^TP + PA + \delta N^TN & PM + \delta N^TD \\ M^TP + \delta D^TN & -\tfrac{1}{\delta}I + \delta D^TD \end{bmatrix} < 0$$

$$\Updownarrow$$

$$P > 0, \quad \begin{bmatrix} A^TP + PA & PM & N^T \\ M^TP & -\tfrac{1}{\delta}I & D^T \\ N & D & -\tfrac{1}{\delta}I \end{bmatrix} < 0 \tag{40}$$

Looking at the lower right block of (40), it is clear that the above condition implies

$$\begin{bmatrix} -\tfrac{1}{\delta}I & D^T \\ D & -\tfrac{1}{\delta}I \end{bmatrix} < 0 \Leftrightarrow 0 > -\frac{1}{\delta}I + \delta D^TD \Leftrightarrow I > \delta^2 D^TD$$

which guarantees **well posedeness** via (39).

Corollary 25 *(BRL) The uncertain CT–LTI system $\dot{x}(t) = [Ax(t) + M(I - \Delta D)^{-1}\Delta N]x(t)$ is robustly stable for any $\|\Delta\| \leq \delta$ if the LMI in (40) is feasible with* $P = P^T > 0$.

4.1.3 First Version of BRL: RECAPITULATION

We "recap" all previous results (including the feed–through versions of BRL) using the variable $\gamma \triangleq \frac{1}{\delta}$ from (30),(31). This makes our notation same to standard MATLAB notation...

IF there exists $P = P^T > 0$ satisfying

$$P > 0, \quad \begin{bmatrix} PA + A^TP + \frac{1}{\gamma}N^TN & PM + \frac{1}{\gamma}N^TD \\ M^TP + \frac{1}{\gamma}D^TN & -\gamma I + \frac{1}{\gamma}D^TD \end{bmatrix} < 0$$

$$\Updownarrow$$

$$P > 0, \quad \begin{bmatrix} PA + A^TP & PM & N^T \\ M^TP & -\gamma I & D^T \\ N & D & -\gamma I \end{bmatrix} < 0 \tag{41}$$

THEN the Uncertain CT–LTI system

$$\dot{x}(t) = [Ax(t) + M(I - \Delta D)^{-1}\Delta N]x(t),$$

is robustly stable for any norm bounded uncertainty Δ satisfying $\sigma_{max}(\Delta) = \|\Delta\| \leq \delta \triangleq \frac{1}{\gamma}$.

Computing the Maximum Allowed Uncertainty is now a "mincx" problem [8]:

$\min \gamma$ over $P > 0$ and $\gamma > 0$ under the LMI constraint (41)

Remark 26 *Recall remark 23 i.e. "A single P proves stability for the whole uncertainty range in system matrix A → Quadratic Stability" and check MATLAB's "quadstab" command (Quadratic stability of polytopic or affine parameter-dependent systems – over the entire parameter range and for arbitrarily fast parameter variations).*

BRL (40) is now rewritten in terms of the (CT-ULTI) System Dynamics (30) in order to make clear its interpretation as a Robust Quadratic Stability Condition. **See Remark 21**. The feed–through term is also included for generalization.

The Uncertain CT–LTI system

$$\dot{x}(t) = [A_0 + M_A(I - \Delta_A D)^{-1}\Delta N_A]x(t)$$

(which is (30) generalized in (38) via the inclusion of the feed–through term), with A_0 being the nominal (known and time invariant) system matrix, is robustly stable for any norm bounded uncertainty Δ_A satisfying $\sigma_{max}(\Delta_A) = \|\Delta_A\| \leq \delta_A \triangleq \frac{1}{\gamma_A}$ if there exists $P = P^T > 0$ satisfying

$$P > 0, \quad \begin{bmatrix} PA_0 + A_0^T P + \delta_A N_A^T N_A & PM_A + \delta_A N_A^T D \\ M_A^T P + \delta_A D^T N_A & -\frac{1}{\delta_A} I + \delta_A D^T D \end{bmatrix} < 0$$

$$\Updownarrow$$

$$P > 0, \quad \begin{bmatrix} PA_0 + A_0^T P & PM_A & N_A^T \\ M_A^T P & -\frac{1}{\delta_A} I & D^T \\ N_A & D & -\frac{1}{\delta_A} I \end{bmatrix} = \begin{bmatrix} PA_0 + A_0^T P & PM_A & N_A^T \\ M_A^T P & -\gamma_A I & D^T \\ N_A & D & -\gamma_A I \end{bmatrix} < 0 \tag{42}$$

Remark 27 *For the "no feed–through case" in (30), i.e. for $\dot{x}(t) = [A_0 + M_A \Delta_A N_A]x(t)$, trivially set $D = 0$. Also recall that the $(1,1)$ element of the LMI (42) "demands" that the nominal matrix A_0 is Hurwitz stable !!!*

4.1.4 Second Version of BRL (no feed–through term)

Several variations of BRL in literature. For example, starting from the bound (33) on the Lyapunov derivative **and selecting the admissible value** $\alpha = \delta^2 > 0$ **instead of** $\alpha = \delta > 0$ **for the positive "tuning variable"** α, the previous inequality now becomes

$$(PM)\Delta N + N^T \Delta^T (M^T P) \leq \delta^2 PMM^T P + N^T N$$

and carrying out the same "math", a sufficient condition for Robust Quadratic Asymptotic Stability of (31) is $P > 0$ along with

$$\left(A^T P + PA + \delta^2 PMM^T P + N^T N\right) < 0$$

$$\Updownarrow$$

$$\left(A^T P + PA + N^T N - PM(-\tfrac{1}{\delta^2} I)^{-1}(PM)^T\right) < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} A^T P + PA + N^T N & PM \\ M^T P & (-\frac{1}{\delta^2} I) \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} PA + A^T P & PM \\ M^T P & -\frac{1}{\delta^2} I \end{bmatrix} - \begin{bmatrix} N^T \\ 0 \end{bmatrix} (-I) \begin{bmatrix} N & 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} PA + A^T P & PM & N^T \\ M^T P & -\frac{1}{\delta^2} I & 0 \\ N & 0 & -I \end{bmatrix} = \begin{bmatrix} PA + A^T P & PM & N^T \\ M^T P & -\gamma^2 I & 0 \\ N & 0 & -I \end{bmatrix} < 0 \tag{43}$$

4.1.5 Second Version of BRL with feed–through term

BRL (43) will now be

- **rewritten** in terms of the (CT-ULTI) System Dynamics in (30) making clearer its interpretation as a Robust Quadratic Stability Condition

- **generalized** by adding a feed–through term (see (38))

Corollary 28 *The system* $\dot{x}(t) = [A_0 + M_A(I - \Delta_A D)^{-1}\Delta N_A]x(t)$ *in (30) is robustly stable for any* $\|\Delta_A\| \leq \delta_A \triangleq \frac{1}{\gamma_A}$ *if there exists* $P = P^T > 0$ *satisfying*

$$P > 0, \quad \begin{bmatrix} PA_0 + A_0^T P + N_A^T N_A & PM_A + N_A^T D \\ M_A^T P + D^T N_A & D^T D - \frac{1}{\delta_A^2} I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$P > 0, \quad \begin{bmatrix} PA_0 + A_0^T P & PM_A & N_A^T \\ M_A^T P & -\frac{1}{\delta_A^2} I & D^T \\ N_A & D & -I \end{bmatrix} = \begin{bmatrix} PA_0 + A_0^T P & PM_A & N_A^T \\ M_A^T P & -\gamma_A^2 I & D^T \\ N_A & D & -I \end{bmatrix} < 0 \tag{44}$$

As before, the lower right block in (44) guarantees **well posedeness** via

$$\begin{bmatrix} \frac{-1}{\delta_A^2} I & D^T \\ D & -I \end{bmatrix} < 0 \Leftrightarrow I > \delta_A^2 D^T D$$

Remark 29 *Compare (44) to (42).*

4.1.6 Second Version of BRL: RECAPITULATION

We "recap" all previous results of the second version of BRL (including the feed–through versions) and using $\gamma \triangleq \frac{1}{\delta}$.

The Uncertain CT–LTI system $\dot{x}(t) = [Ax(t) + M(I - \Delta D)^{-1}\Delta N]x(t)$ is robustly stable for any norm bounded uncertainty Δ satisfying $\sigma_{max}(\Delta) = \|\Delta\| \leq \delta \triangleq \frac{1}{\gamma}$ if there exists $P = P^T > 0$ satisfying any of the four equivalent inequalities below (starting from the "Riccati–like" inequality and ending with the easily memorized (45)):

$$A^T P + PA + N^T N + (PM + N^T D)\left[\gamma^2 I - D^T D\right]^{-1}(M^T P + D^T N) < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} A^T P + PA + N^T N & PM + N^T D \\ M^T P + D^T N & D^T D - \gamma^2 I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} PA + A^T P & PM & N^T \\ M^T P & -\gamma^2 I & D^T \\ N & D & -I \end{bmatrix} < 0$$

$$\Updownarrow$$

$$\begin{bmatrix} I & 0 \\ A & M \\ 0 & I \\ N & D \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & M \\ 0 & I \\ N & D \end{bmatrix} \tag{45}$$

4.1.7 A Congruent Transformation relates the two BRL Versions

Compare the two BRL versions:

- **First Version** is given (in terms of $\gamma = \frac{1}{\delta}$) in (41) (equivalently (42)),

- **Second Version** is given (in terms of $\gamma^2 = \frac{1}{\delta^2}$) in (45) (equivalently (43), (44)).

Starting from the Riccati inequality (43) of Second Version (the one without a feed–through term for easiness of presentation - generalized in (45)) i.e. $A^TP + PA + \delta^2 PMM^TP + N^TN < 0$, we shall constructively show that via a Congruent Transformation one can get inequality (41) of First Version.

Define $Y \triangleq \frac{1}{\gamma}P$. Then

$$(43) \Leftrightarrow A^TP + PA + \delta^2 PMM^TP + N^TN < 0$$
$$\Updownarrow$$
$$A^TP + PA + \frac{1}{\gamma^2}PMM^TP + N^TN < 0$$
$$\Updownarrow$$
$$\gamma\left(A^T(\tfrac{1}{\gamma}P) + (\tfrac{1}{\gamma}P)A\right) + (\tfrac{1}{\gamma}P)MM^T(\tfrac{1}{\gamma}P) + N^TN < 0$$
$$\Updownarrow$$
$$\gamma\left(A^TY + YA\right) + YMM^TY + N^TN < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} \gamma\left(A^TY + YA\right) + N^TN & YM \\ M^TY & -I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} \gamma\left(A^TY + YA\right) & YM & N^T \\ M^TY & -I & 0 \\ N & 0 & -I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\text{congruence: pre- and post- multiply by}$$
$$\begin{pmatrix} \frac{1}{\sqrt{\gamma}}I & 0 & 0 \\ 0 & \sqrt{\gamma}I & 0 \\ 0 & 0 & \sqrt{\gamma}I \end{pmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} A^TY + YA & YM & N^T \\ M^TY & -\gamma I & 0 \\ N & 0 & -\gamma I \end{bmatrix} < 0$$

which is (41) of First Version with $D = 0$ and Y instead of P !!!

4.2 Relating BRL and H_∞ system norm implies a Disturbance Rejection interpretation

Apart from the "Robust Stability interpretation", BRL plays an essential role in **the "Energy Gain" computation of Stable Linear Systems** !!! This BRL "interpretation" is a fundamental result relating Time–Domain and Frequency–Domain (H_∞–norm) system characteristics to the feasibility of a strict LMI (BRL) !!!

- Connection with "Small Gain Theorem"...
- Many variations (KYP,Pos–Real Lemma, etc.)
- The concept of "**Dissipativity**" is behind all this...

Consider the LTI System,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \\ z(t) &= Cx(t) + Dw(t) \\ z &= T(s)w \quad \text{with} \quad T(s) = C(sI - A)^{-1}B + D \end{aligned} \tag{46}$$

and assuming A is (Hurwitz) stable and $x(0) = 0$... can interpret $w(t)$ is an "exogenous disturbance", whose effect on $z(t)$ we wish to analyze (minimize)

- for deterministic "w,z" → "System Gain" notions
- for white noise "w" → asymptotic output variance
- for impulse "w" → output energy

Signal Energy ("L_2") Gain: The Energy Gain of a signal $x(t)$ is

$$\|x(t)\|_2 = \sqrt{\int_0^\infty \|x(t)\|^2 dt}, \text{ with } \|x(t)\|^2 = x^T(t)x(t). \tag{47}$$

System Energy Gain: Starting from $z = T(s)w$ with $T(s) = C(sI - A)^{-1}B + D \in H_\infty$ and assuming that: A is (Hurwitz) stable and $x(0) = 0$, the system gain ("Worst Amplification in any direction") is

$$\|T\|_\infty \triangleq \sup_{0<\|w\|<\infty} \frac{\|z\|_2}{\|w\|_2} = \sup_{0<\|w\|<\infty} \frac{\|Tw\|_2}{\|w\|_2}. \tag{48}$$

Remark 30 *"Small" Energy Gain μ implies "Good" disturbance ($w(t)$) attenuation (rejection) since*

$$\|T\|_\infty < \mu \Leftrightarrow \int_0^\infty z^T z dt < \mu^2 \int_0^\infty w^T w dt, \ \forall w \neq 0 \tag{49}$$

Theorem 31 *Consider the CT LTI system in (46) and assume: A is (Hurwitz) stable and $x(0) = 0$. The following three statements are equivalent:*

- *System's Energy Gain* $\|T\|_\infty < \mu$, *i.e*

$$\sup_{0<\|w\|<\infty} \frac{\|Tw\|}{\|w\|} < \mu \,.$$

- $$\|T\|_\infty = \sup_{\omega} \sigma_{\max}(T(j\omega)) < \mu \,.$$

$$i.e. \;\; T^*(j\omega)T(j\omega) < \mu^2 I \;\; \forall \omega \in R \,.$$

- *There exists a solution* $P = P^T > 0$ *of the LMI*

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \mu^2 I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} PA + A^T P & PB & C^T \\ B^T P & -\mu^2 I & D^T \\ C & D & -I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & -\mu^2 I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix} < 0 \tag{50}$$

Remark 32 *The last three LMIs (50) are actually the BRLs presented in (45) in a "robust stability" setting !!! See section (4.2.1) for proof and section (4.2.2) for an alternative proof...*

4.2.1 Proof of Sufficiency (feasible BRL ⇒ H_∞ norm ≤ μ)

Proof of Sufficiency "feasible BRL ⇒ H_∞ norm ≤ μ" is easy !!! Assuming that BRL (50) holds for the CT LTI system $\dot{x}(t) = Ax(t) + Bw(t)$, $z(t) = Cx(t) + Dw(t)$ in (46) with A (Hurwitz) stable and $x(0) = 0$, the time derivative of $V(x) = x^T Px$, $P = P^T > 0$ along the system trajectories is negative and can be expressed as

$$\begin{aligned} \dot{V}(x(t)) &= x^T[A^T P + PA]x + x^T PBw + w^T B^T Px \\ &= \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0 \end{aligned} \tag{51}$$

Noting that

$$\begin{aligned} z^T z &= (x^T C^T + w^T D^T)(Cx + Dw) \\ &= x^T C^T Cx + x^T C^T Dw + w^T D^T Cx + w^T D^T Dw \end{aligned}$$

the objective $z^T z \le \mu^2 w^T w$ (to be proved) can be written as
$x^T C^T Cx + x^T C^T Dw + w^T D^T Cx + w^T[D^T D - \mu^2 I]w \le 0$ or

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - \mu^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0 \tag{52}$$

Now starting from the "2×2" version of BRL in (50), pre– multiplying by $\begin{bmatrix} x \\ w \end{bmatrix}^T$ and post–multiplying by $\begin{bmatrix} x \\ w \end{bmatrix}$, and using (51), (52), we notice that feasible BRL $\Rightarrow$

$$\underbrace{\begin{bmatrix} x \\ w \end{bmatrix}^T \left\{ \begin{bmatrix} A^TP+PA & PB \\ B^TP & 0 \end{bmatrix} + \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD-\mu^2 I \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix} < 0}_{= \dot{V} + z^Tz - \mu^2 w^T w < 0}$$

i.e. starting from BRL in (50) and using the assumed Lyapunov stability, it was shown that $\dot{V} + z^Tz - \mu^2 w^T w < 0$. Integrating this last inequality from 0 to ∞ while taking into account that

- $V(x(\infty)) = V(0) = 0$ (due to system stability assumption)
- $V(x(0)) = V(0) = 0$ (by zero initial state assumption)

can write

$$\int_0^\infty (\dot{V} + z^Tz - \mu^2 w^T w)dt < 0 \Rightarrow$$
$$V(\infty) - V(0) + \int_0^\infty z^T z dt - \int_0^\infty w^T w < 0 \Rightarrow$$
$$\|z\|_2^2 < \mu^2 \|w\|_2^2 \Rightarrow \frac{\|z\|_2}{\|w\|_2} < \mu$$

i.e. H_∞ norm ("Energy Gain") less than μ !!! (see (48)and (49)).

4.2.2 Alternative Proof of Sufficiency (feasible BRL $\Rightarrow H_\infty$ norm $\leq \mu$)

Here is an alternative proof of Sufficiency including the "mechanics" used for deriving the last LMI in (50).

The time derivative of $V(x) = x^TPx,\ P = P^T > 0$ along the system trajectories is negative and can be expressed as

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}^TPx + x^TP\dot{x} = \dot{x}^TPx + x^TP(Ax+Bw) \\
&= \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} PAx + PBw \\ Px \end{bmatrix} \\
&= \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} PA & PB \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\
&= \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0
\end{aligned}$$

But since $\dot{x} = Ax + Bw = [A \ \ B] \begin{bmatrix} x \\ w \end{bmatrix}$ can write

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x \\ Ax+Bw \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

Using this result into the previous matrix (Lyapunov) inequality

$$\dot{V}(x(t)) = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0 \tag{53}$$

On the other hand, starting from the expression (52) for the objective $z^T z \leq \mu^2 w^T w$ (to be proved) can manipulate the matrix appearing there as follows:

$$\begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - \mu^2 I \end{bmatrix} = \begin{bmatrix} 0 & C^T \\ I & D^T \end{bmatrix} \begin{bmatrix} 0 & -\mu^2 I \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -\mu^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}$$

hence (see (52))

$$z^T z - \mu^2 w^T w = \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -\mu^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \tag{54}$$

Now starting from the "2×2" version of BRL in (50), pre– multiplying by $\begin{bmatrix} x \\ w \end{bmatrix}^T$ and post–multiplying by $\begin{bmatrix} x \\ w \end{bmatrix}$, and using (53), (54), we notice that the expression $\dot{V} + z^T z - \mu^2 w^T w < 0$ arising from BRL, can be expressed as

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \left\{ \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -\mu^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \right\} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

Using "block–diagonal" manipulations, the matrix appearing in the previous matrix inequality can be expressed as

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -\mu^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} =$$

$$\left\{ \begin{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T & \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \end{bmatrix} \right\} \left\{ \begin{bmatrix} 0 & P & I & 0 \\ P & 0 & A & B \\ -\mu^2 I & 0 & 0 & I \\ 0 & I & C & D \end{bmatrix} \right\} =$$

$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ 0 & 0 & -\mu^2 I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix} < 0$$

which is the matrix appearing in the last formulation of BRL in (50).

5 H_∞ Synthesis for Continuous Time Systems

5.1 H_∞ State Feedback Synthesis for CT LTI without Uncertainties

Problem setup:

Open–Loop System (Plant):

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t),\ x(0) = 0\\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t)\end{aligned}$$

State Feedback Controller:

$$u(t) = Kx(t)$$

Closed–Loop System:

$$\dot{x}(t) = \underbrace{(A + B_u K)x(t) + B_w w(t)}_{\triangleq A_{cl}x(t) + B_{cl}w(t)}$$

$$z(t) = \underbrace{(C_z + D_{zu}K)x(t) + D_{zw}w(t)}_{\triangleq C_{cl}x(t) + D_{cl}w(t)}$$

with $A_{cl} \triangleq A + B_u K$, $C_{cl} \triangleq C_z + D_{zu}K$, $B_{cl} = B_w$ and $D_{cl} = D_{zw}$. The transfer function $w \to z$ is clearly $T_{cl}(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl} + D_{cl}$, while the control design objectives are: closed-loop stability **and** $\|T_{cl}\|_\infty < \gamma$ for disturbance attenuation.
(Note the change of symbol from μ into the conventional γ for the attenuation...)

Using BRL (see the version presented in (41)), the above two requirements are equivalent ($\Leftrightarrow$) to the existence of $K \in \Re^{m\times n}$ and $P \in S^n$ such that

$$P > 0,\quad \begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} & C_{cl}^T \\ B_{cl}^T P & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{55}$$

The "Congruence + Change of variables" trick again (set $S = P^{-1}$, $W = KS$)

Matrix Inequality (55) is equivalent ($\Leftrightarrow$) to

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} & C_{cl}^T \\ B_{cl}^T P & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix}\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} A_{cl}P^{-1} + P^{-1}A_{cl}^T & B_{cl} & P^{-1}C_{cl}^T \\ B_{cl}^T & -\gamma I & D^T \\ C_{cl}P^{-1} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{56}$$

Now $B_{cl} = B_w$ and $D_{cl} = D_{zw}$ are constant matrices, whereas the products

$$A_{cl}P^{-1} = AP^{-1} + B_u KP^{-1} \ \text{ and }\ C_{cl}P^{-1} = C_z P^{-1} + D_{zu}KP^{-1}$$

can be transformed using change of variables $S = P^{-1}$, $W = KS$ and hence

$$\begin{aligned} A_{cl}P^{-1} &= AP^{-1} + B_u KP^{-1} = AS + B_u W \\ C_{cl}P^{-1} &= C_z P^{-1} + D_{zu}KP^{-1} = C_z S + D_{zu}W \end{aligned}$$

Substituting into (56) we have the following result [1], [2]:

The CT–LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t), \; x(0) = 0 \\ z(t) &= C_z x(t) + D_{zw}w(t) + D_{zu}u(t) \end{aligned}$$

is stabilizable via state feedback $u(t) = Kx(t)$ such that $\|T_{cl}(s)\|_\infty < \gamma$ **if and only if** there exist $S \in S^n$ (SPD matrix) and $Z \in \Re^{m\times n}$ such that

$$S > 0, \quad \begin{bmatrix} AS + B_u W + SA^T + W^T B_u^T & B_w & SC_z^T + W^T D_{zu}^T \\ B_w^T & -\gamma I_{Nu} & D_{zw}^T \\ C_z S + D_{zu}W & D_{zw} & -\gamma I_{Nu} \end{bmatrix} < 0 \tag{57}$$

If LMI (57) has a feasible solution (in terms of S, W, γ), the SSF control gain $K = WS^{-1}$ stabilizes the closed loop system robustly in the sense of "γ-attenuation".

Remark 33 *Optimal H_∞ control set–up: minimize γ subject to the (convex) LMI constraint (57)...*

An alternative (equivalent) synthesis procedure is available using Finsler's Lemma [3]

The result in (57) can be used as a stepping stone for stability analysis and robust stabilization of uncertain systems with norm bounded uncertainty.

5.2 H_∞ State Feedback Synthesis for CT LTI with Norm Bounded Uncertainties

Problem setup:

Open–Loop System (Plant) with Norm Bounded Uncertainties:

$$\begin{aligned}\dot{x}(t) &= (A+\Delta A)x(t) + B_w w(t) + (B_u + \Delta B_u)u(t), \ \ x(0)=0\\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t)\end{aligned}$$

Norm Bounded Uncertainties:

$$[\Delta A \ \ \Delta B_u] = DF\,[E_a \ \ E_b]\,, \ \ F^T F \leq I$$

State Feedback Controller:

$$u(t) = Kx(t)$$

Closed–Loop System:

$$\begin{aligned}\dot{x}(t) &= (A + B_u K + DF(E_a + E_b K))x(t) + B_w w(t) \stackrel{\triangle}{=} A_{cl}x(t) + B_{cl}w(t)\\ z(t) &= (C_z + D_{zu}K)x(t) + D_{zw}w(t) \stackrel{\triangle}{=} C_{cl}x(t) + D_{cl}w(t)\end{aligned}$$

Design Objective: Stabilization AND γ–attenuation (an H_∞ objective)

Using BRL (see the version presented in (41)) with γ instead of μ, the above two requirements are equivalent ($\Leftrightarrow$) to the existence of $K \in \Re^{m\times n}$ and $P \in S^n$ such that

$$P > 0, \ \ \begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} & C_{cl}^T \\ B_{cl}^T P & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{58}$$

The "Congruence + Change of variables" trick again: $S = P^{-1}$, $W = KS$!!!
Matrix Inequality (58) is equivalent ($\Leftrightarrow$) to

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} & C_{cl}^T \\ B_{cl}^T P & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix}\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

$$\Leftrightarrow \begin{bmatrix} A_{cl}P^{-1} + P^{-1}A_{cl}^T & B_{cl} & P^{-1}C_{cl}^T \\ B_{cl}^T & -\gamma I & D^T \\ C_{cl}P^{-1} & D_{cl} & -\gamma I \end{bmatrix} < 0 \tag{59}$$

Now $B_{cl} = B_w$ and $D_{cl} = D_{zw}$ are constant matrices, whereas the products

$$\begin{aligned} A_{cl}P^{-1} &= AP^{-1} + B_u KP^{-1} + DF(E_a + E_b K)P^{-1} = AS + B_u W + DFE_a S + DFE_b W \\ C_{cl}P^{-1} &= C_z P^{-1} + D_{zu}KP^{-1} = C_z S + D_{zu} Z \end{aligned} \tag{60}$$

were transformed using change of variables $S = P^{-1}$, $W = KS$. When these expressions are used into (59) we have the following equivalent matrix inequality:

$$S > 0,$$

$$\begin{bmatrix} AS + B_u W + SA^T + W^T B_u^T + DF(E_a S + E_b W) + (E_a S + E_b W)^T F^T D^T & B_w & SC_z^T + W^T D_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_z S + D_{zu} W & D_{zw} & -\gamma I \end{bmatrix} < 0 \tag{61}$$

By decomposing the last matrix inequality (61) as

$$\begin{bmatrix} AS+B_uW+SA^T+W^TB_u^T & B_w & SC_z^T+W^TD_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_zS+D_{zu}W & D_{zw} & -\gamma I \end{bmatrix}+$$

$$\begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} (E_aS+E_bW) & 0 & 0 \end{bmatrix} + \begin{bmatrix} (E_aS+E_bW)^T \\ 0 \\ 0 \end{bmatrix} F^T \begin{bmatrix} D^T & 0 & 0 \end{bmatrix} < 0 \tag{62}$$

Inequality (62) is clearly of the form $G+M\Delta N+N^T\Delta^TM^T<0$, with $\Delta\to F$, $M\to\begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$, $N\to\begin{bmatrix} (E_aS+E_bW) & 0 & 0 \end{bmatrix}$ and its "G"–part symmetric. Hence Lemma 5 can be used to transform (62) into the following equivalent "$G+\epsilon MM^T+\frac{1}{\epsilon}N^TN$" matrix inequality (valid for all admissible uncertainties ($F^TF\leq I$)) and $\epsilon>0$. Noting that $\epsilon MM^T=\begin{bmatrix} +\epsilon DD^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, (62) is thus transformed into

$$\begin{bmatrix} (AS+B_uW)+(SA^T+W^TB_u^T)+\epsilon DD^T & B_w & SC_z^T+W^TD_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_zS+D_{zu}W & D_{zw} & -\gamma I \end{bmatrix}+\frac{1}{\epsilon}\begin{bmatrix} (E_a+E_bW)^T \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} (E_a+E_bW) & 0 & 0 \end{bmatrix}<0 \tag{63}$$

which, by Schur Complement is equivalent ($\Leftrightarrow$) to

$$\begin{bmatrix} (AS+B_uW)+(SA^T+W^TB_u^T)+\epsilon DD^T & B_w & SC_z^T+W^TD_{zu}^T & (E_a+E_bW)^T \\ * & -\gamma I & D_{zw}^T & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} \tag{64}$$

If LMI (64) has a feasible solution (in terms of $S,\ W,\ \gamma,\ \epsilon$), the SSF control gain $K=WS^{-1}$ stabilizes the closed loop system robustly in the sense of "γ-attenuation" for all admissible norm bounded uncertainties.

5.3 H_∞ Dynamic Output Feedback Synthesis for CT LTI without Uncertainties - (INCOMPLETE)

Based on [9] and ...

5.4 H_∞ mixed sensitivity design(s)

The theoretical background for the "mixed sensitivity" approach to H_∞ synthesis can be found in [5] chapters 2,5 and 7... A brief recapitulation is depicted in Figures 3, 4, 5, 6 below...

5.4.1 Mixed sensitivity design

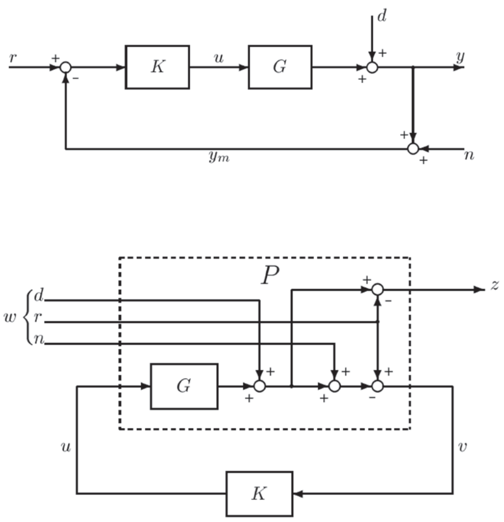
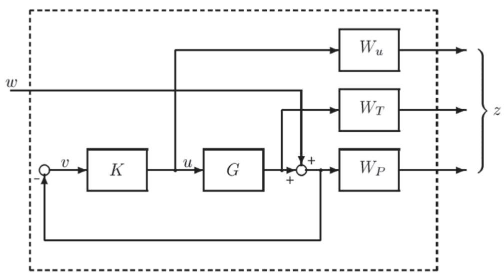


Figure 3: setup for setpoint tracking, disturbance rejection and noise suppression Fig3-14 from [5]



$$\min_K \|N(K)\|_\infty, \quad N = \begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix}$$

Figure 4: "mixsyn" setup for setpoint tracking (Fig3-17 from [5])

5.4.2 Selecting the filters W_p, W_t, W_u

We are interested in S and T:

$$|L(j\omega)| \gg 1 \quad \Rightarrow \quad S \approx L^{-1}; \quad T \approx 1$$
$$|L(j\omega)| \ll 1 \quad \Rightarrow \quad S \approx 1; \qquad T \approx L$$

but in the crossover region where $|L(j\omega)|$ is close to 1, one cannot infer anything about S and T from $|L(j\omega)|$.

Alternative:

Directly shape the magnitudes of closed-loop $S(s)$ and $T(s)$.

In order to enforce specifications on other transfer functions:

$$\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1; \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u KS \end{bmatrix} \tag{2.41}$$

N is a vector and the maximun singular value $\bar{\sigma}(N)$ is the usual Euclidean vector norm:

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u KS|^2} \tag{2.42}$$

The $\mathcal{H}_\infty$ optimal controller is obtained from

$$\min_K \|N(K)\|_\infty \tag{2.43}$$

where K is a stabilizing controller

where K is a stabilizing controller. Let $\gamma_0 = \min_K \|N(K)\|_\infty$ denote the optimal $\mathcal{H}_\infty$ norm. An important property of $\mathcal{H}_\infty$-optimal controllers is that they yield a flat frequency response, that is, $\bar{\sigma}(N(j\omega)) = \gamma_0$ at all frequencies. The practical implication is that, except for at most a factor $\sqrt{n}$, the transfer functions resulting from a solution to (2.79) will be close to γ_0 times the bounds selected by the designer. This gives the designer a mechanism for directly shaping the magnitudes of $\bar{\sigma}(S)$, $\bar{\sigma}(T)$, $\bar{\sigma}(KS)$, and so on. A good

Figure 5: morari-2-22-new (see [5])

The stacking procedure is selected for mathematical convenience as it does not allow us to exactly specify the bounds on the individual transfer functions as described above. For example, assume that $\phi_1(K)$ and $\phi_2(K)$ are two functions of K (which might represent $\phi_1(K) = w_P S$ and $\phi_2(K) = w_T T$) and that we want to achieve

$$|\phi_1| < 1 \quad \text{and} \quad |\phi_2| < 1 \tag{2.77}$$

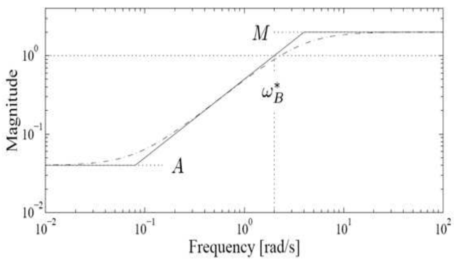
This is similar to, but not quite the same as the stacked requirement

$$\bar{\sigma}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \sqrt{|\phi_1|^2 + |\phi_2|^2} < 1 \tag{2.78}$$

Objectives (2.77) and (2.78) are very similar when either $|\phi_1|$ or $|\phi_2|$ is small, but in the worst case when $|\phi_1| = |\phi_2|$, we get from (2.78) that $|\phi_1| \leq 0.707$ and $|\phi_2| \leq 0.707$. That is, there is a possible "error" in each specification equal to at most a factor $\sqrt{2} \approx 3$ dB. In general, with n stacked requirements the resulting error is at most $\sqrt{n}$. This inaccuracy in the specifications is something we are probably willing to sacrifice in the interests of mathematical convenience. In any case, the specifications are in general rather rough, and are effectively knobs for the engineer to select and adjust until a satisfactory design is reached.

Figure 6: (important explanation from [5] ch.2, p.64)

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}$$



Exact and asymptotic plot of $1/|w_P(j\omega)|$

Typical specifications in terms of S:

1. Minimum bandwidth frequency ω_B^*.
2. Maximum tracking error at selected frequencies.
3. System type, or alternatively the maximum steady-state tracking error, A.
4. Shape of S over selected frequency ranges.
5. Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \leq M$.

Specifications may be captured by an upper bound, $1/|w_P(s)|$, on $\|S\|$.

To get a steeper slope for L (and S) below the bandwidth:

$$w_P(s) = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}$$

Figure 7: "morariFig2-27-2-26" selecting filter W_p (see [5] ch.2, p.61,62

6 Bounded Real Lemma for Discrete Time systems (DT-BRL)

6.1 BRL as a Robust Quadratic Stability Criterion for DT-LTI Uncertain Systems

Consider the "unforced" uncertain DT system

$$x_{k+1} = A(\Delta)x_k = (A_0 + \Delta A)x_k = (A_0 + M_A\Delta_A N_A)x_k \tag{65}$$

Definitions and control objectives similar to the CT case... see also (30) and Remark (21) about temporary notation simplifications...

Remark 34 *Note that (65) can be written as the feedback interconnection of the systems*

$$x_{k+1} = A_0x_k + M_Aw_k, \quad z_k = N_Ax_k, \quad w_k = \Delta z_k \tag{66}$$

Can generalize (65),(66) by including a feed–through term in the "output" equation i.e.

$$z_k = N_Ax_k + Dw_k \tag{67}$$

with the uncertain element $w_k = \Delta z_k$.

Provided that $(I - \Delta_A D)^{-1}$ exists, **a well posedeness requirement**, i.e.

$$(I - \Delta_A D) \text{ nonsingular } \forall\Delta_A \;\Leftrightarrow\; I > \gamma_A^2 D^T D \tag{68}$$

can write

$$w_k = \Delta z_k \Rightarrow w_k = (I - \Delta_A D)^{-1}\Delta_A N_A x_k$$

and the **"LFT" representation of the uncertain DT system** becomes

$$x_{k+1} = [A_0 + M_A(I - \Delta_A D)^{-1}\Delta_A N_A]x_k \tag{69}$$

(Compare with the corresponding CT formulation in (39),(38)...)

Similarly with remark (21) we temporarily omit the "0" subscript from the nominal matrix A_0 (writing simply "A") and the "A" subscripts from M_A, Δ_A, N_A (writing simply M, Δ, N).

Thus the problem is the investigation of robust quadratic stability conditions (equivalently stability + "disturbance attenuation") for the norm bounded uncertain system

$$x_{k+1} = Ax_k + Bw_k, \quad z_k = Cx_k + Dw_k, \quad w_k = \Delta z_k \tag{70}$$

with A (Schur) stable and $x(0) = 0$, $\sigma_{max}(\Delta) \leq \gamma(\overset{\triangle}{=} \frac{1}{\mu})$, z being the controlled output and w the "disturbance".

Theorem 35 *Given system (70) and assuming: A is (Schur) stable and $x(0) = 0$, with $T(z) = C(zI - A)^{-1}B + D \in H_\infty$, the following three statements are equivalent:*

- *System's Energy Gain $\|T\|_\infty < \mu$, i.e*

$$i.e. \quad \sup_{0<\|w_k\|<\infty} \frac{\|Tw\|}{\|w_k\|} < \mu\,. \tag{71}$$

- *There exists a solution* $P = P^T > 0$ *of the LMI*

$$A^TPA - P + C^TC + (A^TPB + C^TD)\left[\mu^2 I - (B^TPB + D^TD)\right]^{-1}(B^TPA + D^TC) < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \mu^2 I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} A^TPA - P & A^TPB + C^TD & C^T \\ B^TPA + D^TC & B^TPB - \mu^2 I & D^T \\ C & D & -I \end{bmatrix} < 0 \tag{72}$$

Proof of Sufficiency: i.e. that "feasible BRL $\Rightarrow$ H_∞ norm $\leq \mu$" is easy !!! Assuming that BRL (72) holds for the DT LTI system in (70), the time derivative of $V(x_k) = x_k^T P x_k,\ P = P^T > 0$ along the system trajectories is negative and can be expressed as

$$\begin{aligned} \Delta V_k &= [Ax_k + Bw_k]^T P [Ax_k + Bw_k] - x_k^T P x_k \\ &= \begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} < 0 \end{aligned} \tag{73}$$

The "disturbance attenuation" interpretation in (71) demands that $z_k^T z_k \leq \mu^2 w_k^T w_k$. Following a procedure analogous to the one in section (4.2.1), we first note that

$$\begin{aligned} z_k^T z_k &= (x_k^T C^T + w^T D^T)(Cx + Dw) \\ &= x_k^T C^T C x_k + x_k^T C^T D w_k + w_k^T D^T C x_k + w_k^T D^T D w_k \end{aligned}$$

and hence the objective $z_k^T z_k \leq \mu^2 w_k^T w_k$ writes as
$x_k^T C^T C x_k + x_k^T C^T D w_k + w_k^T D^T C x_k + w_k^T (D^T D - \mu^2 I) w_k \leq 0$ or

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \mu^2 I \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{74}$$

Now starting from the "2×2" version of BRL in (72), pre– multiplying by $\begin{bmatrix} x_k \\ w_k \end{bmatrix}^T$ and post–multiplying by $\begin{bmatrix} x_k \\ w_k \end{bmatrix}$, and using (73), (74), we notice that feasible BRL $\Rightarrow$

$$\underbrace{\begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \left\{ \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - \mu^2 I \end{bmatrix} \right\} \begin{bmatrix} x_k \\ w_k \end{bmatrix} < 0}_{= \Delta V_k + z_k^T z_k - \mu^2 w_k^T w_k < 0}$$

i.e. starting from BRL in (72) and using the assumed Lyapunov stability, it was shown that $\Delta V_k + z_k^T z_k - \mu^2 w_k^T w_k < 0$. Summing this last inequality from 0 to ∞ while taking into account that (i) $\sum_0^\infty (\Delta V_k) = V_\infty - V_0$, (ii) $V_\infty = V(x(\infty)) = V(0) = 0$ (due to system stability assumption) and (iii) $V_0 = V(x(0)) = 0$ (by zero initial state assumption) can write

$$\sum_0^\infty (\Delta V_k) + \sum_0^\infty (z_k^T z_k) - \mu^2 \sum_0^\infty (w_k^T w_k) < 0 \Rightarrow$$
$$V(\infty) - V(0) + \|z_k\|_2^2 - \mu^2 \|w_k\|_2^2 \Rightarrow$$
$$\|z_k\|_2^2 < \mu^2 \|w_k\|_2^2 \Rightarrow T(z) = \frac{\|z_k\|_2}{\|w_k\|_2} < \mu$$

i.e. H_∞ norm ("Energy Gain") less than μ !!!

6.2 Alternative formulations of the DT BRL

Starting from the "2×2" BRL formulation in (72) can write

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \mu^2 I \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} -P & 0 \\ 0 & -\mu^2 I \end{bmatrix} + \begin{bmatrix} A^TPA + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} -P & 0 \\ 0 & -\mu^2 I \end{bmatrix} + \begin{bmatrix} A^TP & C^T \\ B^TP & D^T \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} PA & PB \\ C & D \end{bmatrix} < 0$$
$$\Updownarrow$$
$$\begin{bmatrix} -P & 0 & A^TP & C^T \\ 0 & -\mu^2 I & B^TP & D^T \\ PA & PB & -P & 0 \\ C & D & 0 & -I \end{bmatrix} < 0 \tag{75}$$

where in the last step the Schur Lemma (16) has been used. Performing now a congruent transformation on (76) via the permutation matrix

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

can finally express alternatively the DT–BRL in (72) as

$$\begin{bmatrix} -P & A^TP & 0 & C^T \\ PA & -P & PB & D^T \\ 0 & B^TP & -\mu^2 I & 0 \\ C & 0 & D & -I \end{bmatrix} < 0 \tag{76}$$

Note that the lower left block reflects the well posedeness requirement in (68) i.e. $I > \mu^2 D^T D$.

6.3 A note on the Controllability and Observability of Discrete–Time Systems

INCOMPLETE

See [10] pages 394 – 395

7 The scaling trick for H_∞ Analysis & Synthesis of Uncertain Linear CT & DT Systems

The scaling trick was extensively studied in the 90's for both CT and DT uncertain (norm bounded) linear systems. The most generic presentation can be found in [11] although other useful approaches can also be found in [12] and...

The trick consists into forming an "auxiliary" system without uncertainty (though depending on the matrices appearing in the norm bounded uncertain parts of the system matrices) which is equivalent "in the BRL sense" with the original uncertain system. This equivalence (concerning the BRL of the two systems –original uncertain and "auxiliary" without uncertainty– also entails that an H_∞ controller designed for the "auxiliary" system without uncertainty, is an H_∞ controller for the original uncertain system. The presentation below is a simplified version of the results in [11].

7.1 H_∞ control for uncertain Continuous–Time systems

Consider the uncertain linear system ("Original-Uncertain-Forced-System")

$$\begin{aligned} \dot{x}(t) &= A_\Delta x(t) + B_w w(t) + B_u u(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \end{aligned} \tag{77}$$

and its unforced version

$$\begin{aligned} \dot{x}(t) &= A_\Delta x(t) + B_w w(t), \quad x(0) = 0 \\ z(t) &= C_z x(t) + D_{zw} w(t) \end{aligned} \tag{78}$$

with

$$A_\Delta = A + \Delta A, \quad \Delta A = M_a F N_a, \quad F^T(t)F(t) \le I, \quad \forall t \tag{79}$$

Consider now the following "auxiliary" (scaled) unforced system without uncertainty (for simplicity assume $D_{zw} = 0$)

$$\begin{aligned} \dot{x}_a(t) &= A x_a(t) + \begin{bmatrix} \gamma^{-1} B_w & \varepsilon M_a \end{bmatrix} w_a(t), \quad x_a(0) = 0 \\ z_a(t) &= \begin{bmatrix} C_z x(t) \\ \frac{1}{\varepsilon} N_a \end{bmatrix} x_a, \quad \varepsilon > 0 \end{aligned} \tag{80}$$

The following Theorem establishes an equivalence between the scaling

Theorem 36 *The uncertain system (78) is quadratically stable with H_∞ disturbance attenuation $\gamma > 0$ if and only if for some $\varepsilon > 0$ the scaled auxiliary system (80) is stable with unitary H_∞ disturbance attenuation.*

The idea is that starting from the BRL corresponding to a unitary H_∞ disturbance attenuation for the "auxiliary" (scaled) unforced system in (80) it is possible **using equivalences** to come up with the BRL that corresponds to a H_∞ disturbance attenuation $\gamma > 0$ for the uncertain system (78).

The proof needs the following version of Lemma 5.

Lemma 37 *Given matrices G, M, N of compatible dimensions with G symmetric, the inequality*

$$G + M\Delta N + N^T \Delta^T M^T < 0$$

holds for all Δ *satisfying* $\Delta^T\Delta \leq I$ ***if and only if*** *(*$\Leftrightarrow$*) there exists a constant* $\epsilon > 0$ *such that*

$$G + \epsilon^2 M M^T + \frac{1}{\epsilon^2} N^T N < 0 \quad \Leftrightarrow \quad \begin{bmatrix} G & \begin{pmatrix} \frac{1}{\epsilon}N^T & \epsilon M \end{pmatrix} \\ \begin{pmatrix} \frac{1}{\epsilon}N \\ \epsilon M^T \end{pmatrix} & -I_{2n} \end{bmatrix} < 0$$

$$\Leftrightarrow \quad \begin{bmatrix} G & \begin{pmatrix} \epsilon M & \frac{1}{\epsilon}N^T \end{pmatrix} \\ \begin{pmatrix} \epsilon M^T \\ \frac{1}{\epsilon}N \end{pmatrix} & -I_{2n} \end{bmatrix} < 0$$

Proof of Theorem 36: The BRL corresponding to a unitary H_∞ disturbance attenuation for the "auxiliary" (scaled) unforced system in (80) is

$$\begin{bmatrix} A^T P + PA & PM & N^T \\ M^T P & -\gamma I & D^T \\ N & D & -\gamma I \end{bmatrix} < 0 \tag{81}$$

7.2 H_∞ control for uncertain Discrete–Time systems

The following Lemmas are cited in and concern the Robust H_∞ control of linear discrete-time systems with norm-bounded time-varying uncertainty.

Consider the unforced uncertain linear system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + H_1 w(k), \\ y(k) &= Cx(k) + Du(k) + H_2 w(k) \\ z(k) &= E_1 x(k) + E_2 u(k) \end{aligned} \tag{82}$$

with $E_2^T E_2 > 0$

Lemma 38

2-BE CONTINUED...

8 MATLAB code for various Robust and H_∞ analysis & synthesis approaches

Remark 39 *MATLAB's Robust Control Toolbox Manual uses the symbol γ as system's H_∞ norm (see e.g. "hinfsyn" command)...*

...whereas in this lecture, system's H_∞ norm is denoted by μ !!!

see for example page $8-7$ (209/675) of "Robust Control Toolbox User's Guide September 2007"...

...see also pages $5-28$ (118/130) of "Getting Started with Robust Control Toolbox ver 3.3 R2007b" - chapter Interpretation of H-Infinity Norm...

8.1 MATLAB code-1: Robust CT SSF Synthesis for Uncertain CT Unstable Sys

Consider the nominal open–loop unstable "benchmark" system $G(s) = \frac{1}{s^2}$ with state–space description (double integrator in controllable canonical form)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

presented in Appendix 16. The uncertainties D_a, $E_a(\alpha)$, $E_b(\beta)$ used in the simulations are:

$$D_a = \begin{bmatrix} 50 & 50 \end{bmatrix}, E_a = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad E_b = 10$$

with F_α being a uniform random variable taking values in the interval $(-1,\ 1...$ i.e. $|F_\alpha| < 1$. The initial conditions are $X_{init} = \begin{pmatrix} 10 & -10 \end{pmatrix}$.

The proposed LMI (22) in section 3.4 yields a state feedback gain $K_{sf} = \begin{pmatrix} -0.1000 & -0.1000 \end{pmatrix}$. Figure 8 presents the Z.I.R state response and the control signal.

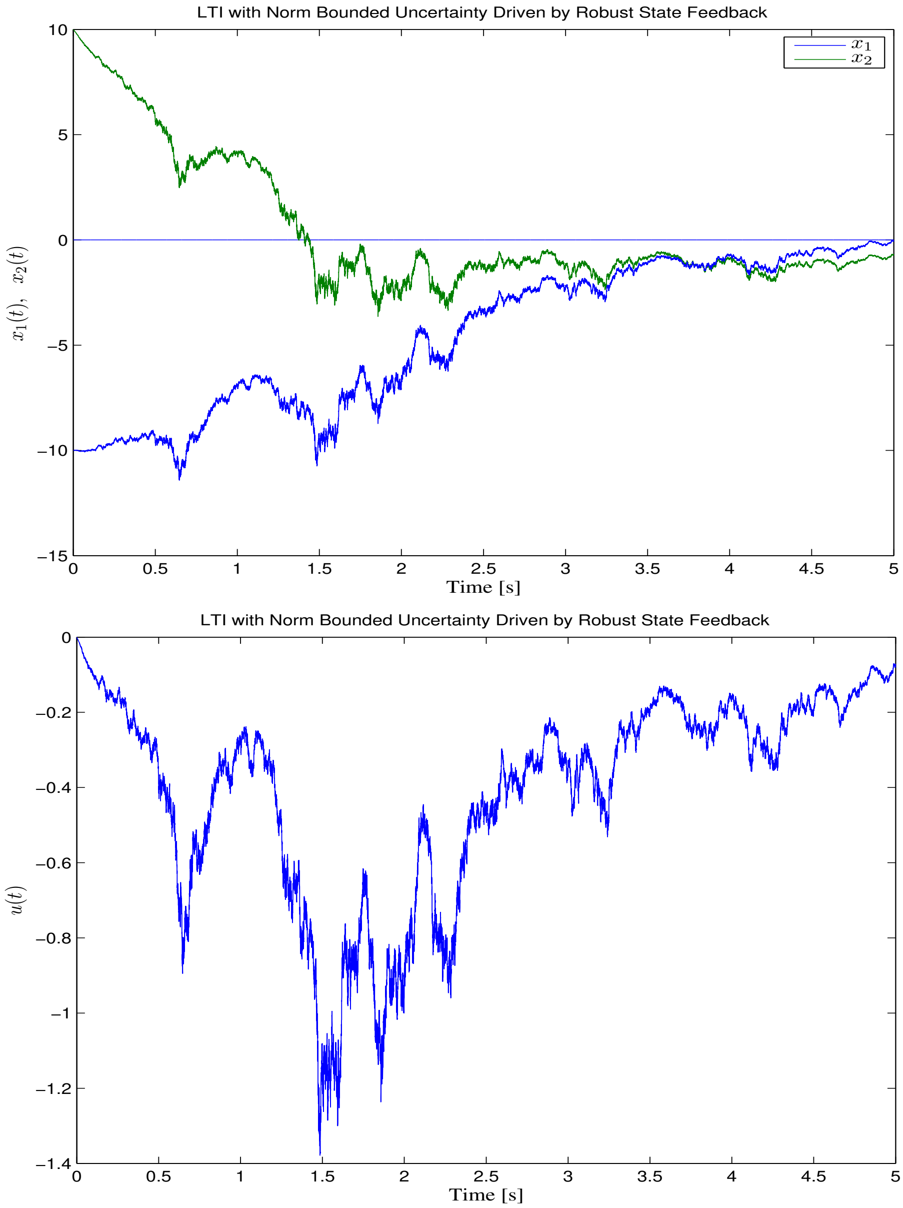


Figure 8: Uncertain SYS0 (double integrator) State vector and control action

8.2 MATLAB code-2: minimum H_∞ system norm γ via BRL–LMI + "mincx"

```
%-------------------------------------------------------------
% Example of an Linear objective minimization problem (H-infinity norm)
%  disp('Leonidas DRITSAS 26Apr08') G = rss(5,2,2); [A,B,C,D] = ssdata(G);
%-------------------------------------------------------------

setlmis([ ])
%-----         Specfiy the matrix variables
P = lmivar(1,[size(A,1) 1]); gamma = lmivar(1,[1 1]);
%-----      New LMI
HinfLMI = newlmi   ;

%-----      Only terms above the diagonal need to be specified
lmiterm([HinfLMI 1 1 P],1,A,'s')  % AP + P'A
lmiterm([HinfLMI 1 2 P],1,B)      % PB
lmiterm([HinfLMI 1 3 0],C')       % C'
lmiterm([HinfLMI 2 2 gamma],-1,1) %-gamma.I
lmiterm([HinfLMI 2 3 0],D')       %D'
lmiterm([HinfLMI 3 3 gamma],-1,1) %-gamma.I
Ppos = newlmi                  % New LMI
lmiterm([Ppos 1 1 P],-1,1)     % P > 0
LMIsys = getlmis;               % Obtaining the system of LMIs


c = mat2dec(LMIsys,zeros(size(A,1),size(A,1)),1);
options = [1e-5,0,0,0,0]; % Relative accuracy of solution

%-----      Solving the minimization problem  --> use  mincx
[copt,xopt] = mincx(LMIsys,c, options);

%-----      Obtaining the optimal P and the optimal gamma
Popt = dec2mat(LMIsys,xopt,P); gammaopt =
dec2mat(LMIsys,xopt,gamma);

%-----      display  results
disp('gammaopt  is...');  disp(gammaopt) disp('Popt  is...');
disp(Popt)

%-----       Verify definiteness of matrix  P  --------------
disp('eigenvalues of Popt are...');  disp(eig(Popt)) [R1,p1] =
chol(Popt); if p1==0
    disp('CHOLESKY VERIFIES  THAT Popt is Pos-Def ');
else
    disp('CHOLESKY SAYS THAT Popt is NOT Pos-Def --> INSTABILITY !!!');
end
%-----      END
```

8.3 MATLAB code-2: Robust SSF Synthesis via LMI for uncertain CT system

The following function implements the LMI synthesis procedure (22) presented in section 3.4 for the robust state feedback stabilization of CT LTI systems with norm–bounded uncertainty ($\dot{x} = (A + \Delta A)x(t) + (B + \Delta B)u(t)$). The LMI (22) whose feasibility yields the desired state feedback gain is presented below as a reminder.

$$\begin{bmatrix} (AS + BW) + (SA^T + W^T B^T) + \epsilon DD^T & (E_a S + E_b W)^T \\ (E_a S + E_b W) & -\epsilon I_n \end{bmatrix} < 0$$

```
%------------------------------------------------------------
%    function "fcn_solve_UNC_CT_LMI"
%         DRITSAS  2009/2010
%------------------------------------------------------------
% disp('function "fcn_solve_UNC_CT_LMI" Computes  Robust Static
% State Feedback  u=+Kx ')

function [Sopt,Wopt,epsilonopt]=fcn_solve_UNC_CT_LMI(A,B,Ea,Eb,D)

[Nx Nu] = size(B);

%-------------------   LMI SETUP  ----------------------
setlmis([]);

%---- define S = inv(P)  as  SYMMETRIC
S=lmivar(1, [Nx , 1]); %% SYMMETRIC - BLOCK DIAGONAL - FULL BLOCK
%---- define W
W  =lmivar(2, [Nu , Nx]); %W=lmivar(2, [1 , 2]);  %W  is 1xNx  full rectangular
%---- define SCALAR   epsilon
epsilon=lmivar(1, [1 1] );

%-------------------------------------------------------
% POSITIVE DEFINITENESS OF S, epsilon
%-------------------------------------------------------
Sposdef=newlmi;  % newlmi Sposdef = POSITIVE DEFINITENESS OF S
lmiterm ( [Sposdef 1 1 S ],-1,1); %  -S < 0

%disp(' Omitting the constraint on EPSILON gives diff results !!!'); pause
EPSILONposdef=newlmi;    % epsilon > 0
lmiterm([EPSILONposdef 1 1 epsilon],-1,1);  % -epsilon < 0


%-------------------------------------------------------
%      newlmi LDRI
%-------------------------------------------------------
LDRI=newlmi;
```

```
%---  1 1 -->
lmiterm([LDRI 1 1 S],A,1,'s'); %%'s' --> AS + S'A'
lmiterm([LDRI 1 1 W],B,1,'s'); %%'s' --> BW + W'B'
lmiterm([LDRI 1 1 epsilon],D,D') % epsilon*DD'


%---  1  2 --> S'*Ea' + W'*Eb'
%----    BEWARE !! THE VARIABLEs "S" & "W" are TRANSPOSED !! -----
% TERMID(4) =  0  ->  constant term
% TERMID(4) =  X  ->  variable term A*X*B
% TERMID(4) = -X  ->  variable term A*X'*B % where X is the variable identifier in LMIVAR
disp('Recall: TERMID(4) = -X  ->  variable term A*Xtranspose*B ')
lmiterm([LDRI 1 2 -S], 1, Ea');  %% S'*Ea'
lmiterm([LDRI 1 2 -W], 1, Eb' ); %% W'*Eb'

%---  2 2 --> -epsilon*I
disp('The  IDENTITY/ZERO matrix is square n x n ')
lmiterm([LDRI 2 2 epsilon],-1,1) % -epsilon*I

%---   getlmis
lmisys=getlmis;

%-------------------------------------------------------------------
%   LMI   SOLVE
%-------------------------------------------------------------------
disp('LDRI: I increased  OPTIONS(2)= max. num of iterations into
1500 ! ')
% OPTIONS(2): max. number of iterations (Default=100)
% OPTIONS(4): when set to an integer value  L > 1, forces termination when  t  has not
%      decreased by more than 1 over the last L iterations  (Default = 10).
options = [0,1500,-1,150,0] ;

% TARGET optional:  target for TMIN.  The code terminates as
%                 soon as   t < TARGET      DEFAULT = -1e5
target = -1e-10;   %target=0

%--------------- FEASP   --------------------------------------
[tmin,xfeas]=feasp(lmisys,options,target);

disp('tmin=');disp(tmin)

fprintf('\n\n');
fprintf('*******************************************************');
disp('     ***  LDRI:  check tmin                          ***');
disp('*** verify pos-definiteness  of the solution S ***');
fprintf('*******************************************************');

if tmin < 0;
    disp('LDRI_11May09: feasp LMI Feasible  SINCE  tmin < 0');
```

```
else
    error('LDRI_11May09: feasp LMI is not Feasible SINCE  tmin > 0 ');
end

%---------------------------------------------------------------------------
%   optimal  S = inv(P)
%---------------------------------------------------------------------------
Sopt = dec2mat(lmisys,xfeas,S)

disp('eigenvalues of optimal S are...');  disp(eig(Sopt))

[R1,p1] = chol(Sopt); if p1==0
    disp('--- CHOLESKY VERIFIES  THAT optimal S is Pos-Def --- ');
else
    error('-- CHOLESKY VERIFIES  THAT optimal S is NOT Pos-Def');
end

%---------------------------------------------------------------------------
%   optimal  X1 --> X=X1'  &  K=X*inv(Sopt)
%---------------------------------------------------------------------------
Wopt = dec2mat(lmisys,xfeas,W)  %% Xopt=X1opt'

%---------------------------------------------------------------------------
%   optimal  epsilon
%---------------------------------------------------------------------------
epsilonopt=dec2mat(lmisys,xfeas,epsilon)


%-----  end   -----------------------------------------------
```

8.4 MATLAB code-3: Robust SSF Synthesis via LMI for uncertain DT system (INCOMPLETE)

The following function implements the LMI synthesis procedure presented in section 10.4 for the robust state feedback stabilization of DT LTI systems with norm–bounded uncertainty

$$x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k$$

$$[\Delta A \;\; \Delta B] = DF\,[E_a \;\; E_b]\,, \;\; F^T F \leq I$$

The LMI (170) whose feasibility (in terms of a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrix $S = P^{-1} \in \Re^{n\times n}$) yields the desired state feedback gain ($u_k = WS^{-1}x_k = Kx_k$) is presented below as a reminder.

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & 0 \\ (AS + BW)^T & -S & (E_a S + E_b W)^T \\ 0 & (E_a S + E_b W) & -\epsilon I \end{bmatrix} < 0$$

```
function [Sopt,Wopt,epsilonopt] =fcn_solve_UNC_DT_LMI(A,B,Ea,Eb,D)

%-----  DRITSAS  2009/2010
disp('function "fcn_solve_UNC_DT_LMI" Computes  Robust Static
State Feedback  u=+Kx ')

[Nx Nu] = size(B);
%-------------------   LMI SETUP  -------------------------------
setlmis([]);

%---- define S = inv(P)  as  SYMMETRIC
S=lmivar(1, [Nx , 1]); %% SYMMETRIC - BLOCK DIAGONAL - FULL BLOCK
%---- define W
W  =lmivar(2, [Nu , Nx]); %W=lmivar(2, [1 , 2]);  %W  is 1xNx  full rectangular
%---- define SCALAR   epsilon
epsilon=lmivar(1, [1 1] );

%-------------  newlmi Sposdef = POSITIVE DEFINITENESS OF S
Sposdef=newlmi;
lmiterm ( [Sposdef 1 1 S ],-1,1) %  -S < 0
%------------------------ POSITIVE SCALAR epsilon > 0
%disp('    Omitting the constraint on EPSILON gives diff results !!! ')
EPSILONposdef=newlmi;    %%
lmiterm([EPSILONposdef 1 1 epsilon],-1,1);% -epsilon < 0

%-------------------------------------------------------
%      newlmi LDRI  =  THEOREM
%-------------------------------------------------------
LDRI=newlmi;

%-------- 1 1
lmiterm([LDRI 1 1 S],-1,1); % -S
lmiterm([LDRI 1 1 epsilon],D,D') % epsilon*DD'
```

```
%-------- 1  2 --> A*S + B*W
lmiterm([LDRI 1 2 S], A, 1); %% A*S
lmiterm([LDRI 1 2 W], B, 1 ); %% B*W
disp('The three ZERO matrices in the first row are square n x n ')
%-------- 1  3 -->
lmiterm([LDRI 1 3 0],zeros(Nx));

%--------  2 2 --> -S
lmiterm([LDRI 2 2 S], -1,1);
%%%   2 3 --> BEWARE !! THE VARIABLEs "S" & "W" are TRANSPOSED
% TERMID(4) =  0  ->  constant term
% TERMID(4) =  X  ->  variable term A*X*B
% TERMID(4) = -X  ->  variable term A*X'*B
% X is the variable identifier returned by LMIVAR
disp(' TERMID(4) = -X  ->  variable term A*Xtranspose*B ')
lmiterm([LDRI 2 3 -S], 1, Ea'); %% S'*Ea'
lmiterm([LDRI 2 3 -W], 1, Eb'); %% W'*Eb'

%--------   3 3 --> -epsilon*I
lmiterm([LDRI 3 3 epsilon],-1,1) % -epsilon*I

%--------------------  getlmis
lmisys=getlmis;

%----------------------------------------------------------------------------
%   LMI   SOLVE
%----------------------------------------------------------------------------
disp('LDRI: I increased  OPTIONS(2)= max. number of iterations
into 1500 !!! ')
% OPTIONS(2): max. number of iterations (Default=100)
% OPTIONS(4): when set to an integer value  L > 1, forces termination when  t
%  has not  decreased by more than 1 over the last L iterations (Default = 10).
options = [0,1500,-1,150,0] ;

% TARGET    optional:  target for TMIN.  The code terminates as
%                soon as   t < TARGET      DEFAULT = -1e5
target = -1e-10;   %target=0

%-------------- FEASP ---------------------------------------
[tmin,xfeas]=feasp(lmisys,options,target); disp('tmin=');
disp(tmin)

if tmin < 0;
     fprintf('\n'); fprintf('***********************************'); fprintf('\n');
     disp('LDRI_11May09: feasp LMI Feasible  SINCE  tmin < 0');
     fprintf('***********************************'); fprintf('\n');
else
     fprintf('\n'); fprintf('*************** LMI not Feasible ******************');
     fprintf('\n');
     error('LDRI_11May09: feasp LMI is not Feasible SINCE  tmin > 0 ');
```

```
     fprintf('***********************************'); fprintf('\n');
end

fprintf('\n\n');
fprintf('*****************************************************************');
disp('  *** get the solutions verify pos-definiteness  of S ***');
fprintf('*****************************************************************');
%----------------------------------------------------------------------------
%   optimal  S = inv(P)
%----------------------------------------------------------------------------
Sopt = dec2mat(lmisys,xfeas,S)
%---------------
disp('eigenvalues of optimal S are...');  disp(eig(Sopt))
%---------------
[R1,p1] = chol(Sopt); if p1==0
    disp('--- CHOLESKY VERIFIES  THAT optimal S is Pos-Def --- ');
else
    error('-- CHOLESKY VERIFIES  THAT optimal S is NOT Pos-Def');
end
%----------------------------------------------------------------------------
%   optimal  W   &  K=W*inv(Sopt)
%----------------------------------------------------------------------------
Wopt = dec2mat(lmisys,xfeas,W)
%----------------------------------------------------------------------------
%   optimal  epsilon
%----------------------------------------------------------------------------
epsilonopt=dec2mat(lmisys,xfeas,epsilon)
```

8.4.1 Numerical Result: Robust DT SSF Synthesis for Uncertain CT Unstable Sys

Consider the nominal open–loop unstable "benchmark" system $G(s) = \frac{0.1}{s^2+0.1s}$ with state–space description (controllable canonical form)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 0.1 & 0 \end{bmatrix} x(t)$$

presented in Appendix 16. The nominal system is discretized with $h = 0.1$s, yielding

$$A_d = \begin{bmatrix} 1 & 0.0995 \\ 0 & 0.9900 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0050 \\ 0.0995 \end{bmatrix}$$

The norm bounded discrete–time uncertainties $D,\ E_a(\alpha),\ E_b(\beta)$ used in the simulations obey the following structure:

$$D = sysfactor * [-1;\ 1]; Ea = sysfactor * [-1\ 1]; Eb = sysfactor * 1;$$

hence

$$\Delta A = (sysfactor)^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} f, \quad \Delta B = (sysfactor) \begin{bmatrix} -1 \\ 1 \end{bmatrix} f, \ |f| < 1$$

The LMI (170) was found infeasible for $sysfactor > 0.35$ whereas for $sysfactor = 0.35$, the robustly stabilizing state feedback gain was computed as

$$K = \begin{bmatrix} -14.1930 & -9.6521 \end{bmatrix}$$

yielding a zero–input-response shown below in Figure 9 when the initial conditions are $X_{init} = \begin{bmatrix} 10 & -10 \end{bmatrix}$.

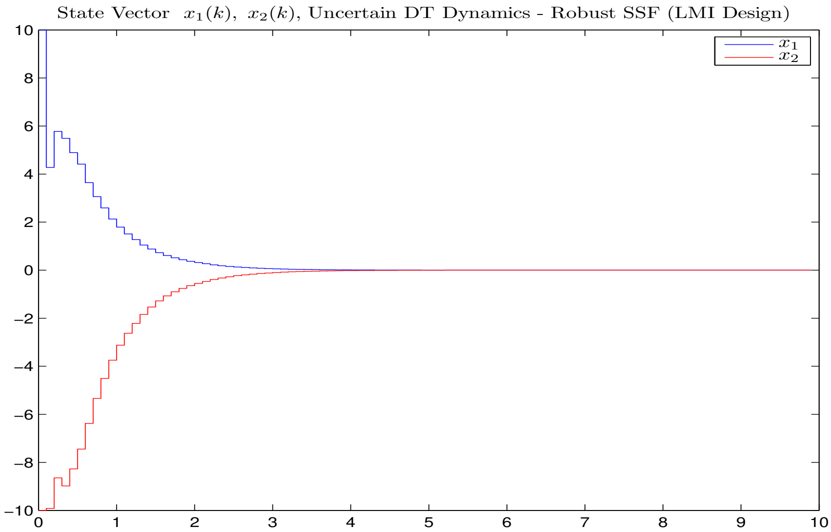


Figure 9: State vector of the Uncertain SYS7 ($sysfactor = 0.35$)

8.5 MATLAB code-4: H_∞ SSF synthesis (INCOMPLETE)

8.6 MATLAB code-5: Using "hinfsyn/mixsyn" commands for H_∞ synthesis (INCOMPLETE)

```
clear all; clc ; close all fig=5;
disp('=================================================================')
disp('LDRI 09Sep10 + 27Dec07  -->   Using mixsyn...') ;
%---
disp('This is a version including  Wt (along with Wp,Wu) ')
disp('=================================================================')

disp('=======  SIMULINK QUANTITIES ============')
%---
Unif_Rand_Number_Amplit=1;
y_ref = 1%0
%---- step DISTURBANCE
step_final_value_disturb = +0.5*y_ref;
step_time_disturb = 20 ;%% Starting Time for DISTURBANCE "step"
%---- pulse DISTURBANCE
pulse_disturb_amplitude = +0.5*y_ref; %+1;
pulse_disturb_period = 0.50;
disp('=================================================================')

s=tf('s');
disp('=================================================================')
disp('  Default PLANT Gtf = (200) / (10s + 1 ) (0.05s + 1)^2 &
Zero Init_Conditions  ')
disp('=================================================================')
Gtf = (200)/( (10*s+1)*(0.05*s+1)^2  )


pole(Gtf) ; tzero(Gtf)
Gss = ss(Gtf)
[G_Nx G_Nu] = size(Gss.b)
xinit_Plant = zeros(G_Nx,1)

%WEIGHT Wp = (s/M + wb)/ (s +wb*A) ,  Wu = 1
disp('=================================================================')
disp('  Sensitivity weight Wp== (s/M+wb)/(s+wb*A) and 1/Wp are
BOTH Stable          ')
%----
disp('NOTE: a value of A = 0 in Wp would ask for integral action
in the controller')

wb=10;   % Closed loop bandwidth, wb
M=1.5;   % desired bound on hinfnorm(S) & hinfnorm(T)
A=10^-4; % desired disturbance attenuation inside bandwidth

disp('*** Wp = Sensitivity weight (zpk  FORMAT)')
Wp = (s/M+wb)/(s+wb*A); Wp_zpk=zpk(Wp)  % Sensitivity weight
```

```
%-----------------
disp('*** invWp = inverse of Sensitivity weight Wp (zpk  FORMAT)')
invWp = 1/Wp ;  invWp_zpk=zpk(invWp)

   if isstable(Wp)==1
       disp('*** Wp stable'); %disp('  *** pole(Wptf) ***  '); pole(Wp)
   else
       disp('  *** pole(Wptf) ***  '); pole(Wp)
       error('*** Wp UNSTABLE')
   end
   %------------------
   if isstable(invWp)==1
       disp('*** invWp stable');
   else
       disp('  *** pole(invWp) ***  '); pole(invWp)
       error('*** invWp UNSTABLE')
   end

disp('=================================================================')
disp(' Wt=1/Wi = weight on T (noise attenuation at high-freq) ')
Wt=(s+wb/M)/(A*s+wb); Wt_zpk=zpk(Wt)% Complementary sensitivity weight
disp(' invWt = inverse of Complementary Sensitivity weight Wt(zpk
FORMAT)')
%----
invWt=1/Wt; invWt_zpk=zpk(invWt)
%----
disp(' pole(Wt)  ');pole(Wt)
%----
disp('  *** pole(invWt) *** ');pole(invWt)

figure(fig); fig=fig+1;
bodemag(invWp,'b', invWt,'g')
title('Bode-mag of 1/Wp, 1/Wt') hh=legend('$|1/W_{p}(j\omega)|$',
'$|1/W_{t}(j\omega)|$',12);
set(hh,'Interpreter','latex','FontName', 'Times New
Roman','fontsize',12)

disp('==================================================')
%---
disp(' Control ("Input Sensitivity) weight Wu=1 ')
Wu=1;                   % Control weight

disp('==================================================')

disp(' mixsyn syntax ')

disp(' [K,CL,GAM,INFO]=
mixsyn(G,W1,W2,W3,KEY1,VALUE1,KEY2,VALUE2,...) ')

disp('is equivalent to
```

```
[K,CL,GAM,INFO]=hinfsyn((G,W1,W2,W3),KEY1,VALUE1,KEY2,VALUE2,...)')

disp('NOTE: This  version includes  Wt  (along with Wp,Wu) ')
disp('===================================================')

% In our case  CL = [W1*S; W2*K*S; W3*T] = N = [Wp1*S; Wu2*K*S; W3=empty]
[Kmixsyn_ss, N, GAMMA_mixsyn, INFO] = mixsyn(Gss,Wp,Wu,Wt);

disp('* VERIFY that "mixsyn = hinfsyn(augw(Gss,Wp,Wu,Wt))"  *');

[Khinfsyn_ss, ghinf, gopt_hinfsyn] = hinfsyn(augw(Gss,Wp,Wu,Wt))


disp('===================================================')
disp('LDRI:   THE  Hinf-mixsyn CONTROLLER is returned by mixsyn in
"ss"  format')
%----
disp('    Set Zero Initial-Conditions Hinf-mixsyn CONTROLLER ')
disp('===================================================')
Kmixsyn_ss   % THE  Hinf-mixsyn CONTROLLER
[K_Nx K_Nu] = size(Kmixsyn_ss.b) xinit_Kss = zeros(K_Nx,1)

disp('== Kmixsyn_tf = Hinf_mixsyn CONTROLLER in TF FORMAT =====')
Kmixsyn_tf = tf(Kmixsyn_ss) % K = tf(Kss);
disp('== Kmixsyn_zpk = Hinf_mixsyn CONTROLLER in zpk FORMAT=====')
Kmixsyn_zpk = zpk(Kmixsyn_tf)
%----
disp('  *** poles  of  Hinf_mixsyn CONTROLLER *** ')
pole(Kmixsyn_zpk)

fprintf('\n\n');
disp('===================================================')
disp('LDRI: CALCULATE TRANSFER-FUNCTIONS  L, S, T and their Hinf
NORMS  ');
disp('===================================================')
L = Gtf*Kmixsyn_tf  ;  % L2= G*Kmixsyn_ss ;   % also   valid !!!!
%-- Sensitivity
S= inv(1+L) ; %S=minv(madd(1,L));
%-- Complementary sensitivity
T=1-S ; Tzpk=zpk(T)
%---
disp(' poles  of T(s) with Hinf_mixsyn CONTROLLER ');
pole(Tzpk)
%-- KS
R=Kmixsyn_ss*S ;

disp('====  Compare GAMMA_mixsyn <==> N_peak= norm(N,inf) ===== ')
S_peak = norm(S,inf)  %
T_peak = norm(T,inf)  %
N_peak = norm(N,inf)  %
```

```
fprintf('\n\n');
disp('==================================================')
disp('LDRI:        SIGMA  PLOTS... ');
disp('==================================================')
figure(fig); fig=fig+1; sigma(S,'b', T,'g', GAMMA_mixsyn/Wp,'r-.',
GAMMA_mixsyn*Gtf/ss(Wu),'k-.')
hh=legend('$\sigma(S)$','$\sigma(T)$' ,'$\gamma/W_{p}$',
'$\gamma*G/W_{u}$' , 12)
%---
set(hh,'Interpreter','latex','FontName', 'Times New
Roman','fontsize',12)

%-----------------------------------------------
%              12 = BODE of S, N_peak/Wp
%-----------------------------------------------
figure(fig); fig=fig+1;
sigma(S, N_peak/Wp,'r-.'); %sigma(S,1/Wp,'-.');
title('|S| and \gamma/|W_{p}| using  "sigma" ');

hh=legend('$|S(j\omega)|$', '$\gamma/|W_{p}(j\omega)|$',12)
set(hh,'Interpreter','latex','FontName', 'Times New
Roman','fontsize',12)

%-----------------------------------------------
%              13 = BODE of T, N_peak/Wt
%-----------------------------------------------
figure(fig); fig=fig+1;
sigma(T, N_peak/Wt,'r-.'); %sigma(S,1/Wp,'-.');
title('|T| and \gamma/|W_{t}| using  "sigma" ');

hh=legend('$|T(j\omega)|$', '$\gamma/|W_{t}(j\omega)|$',12)
set(hh,'Interpreter','latex','FontName', 'Times New
Roman','fontsize',12)

%-----------------------------------------------
%              14 = step response of T
%-----------------------------------------------
figure(fig); fig=fig+1; step(T,3); xlabel('Time');ylabel('y');
title('Tracking a STEP:     the step response of T     ');
```

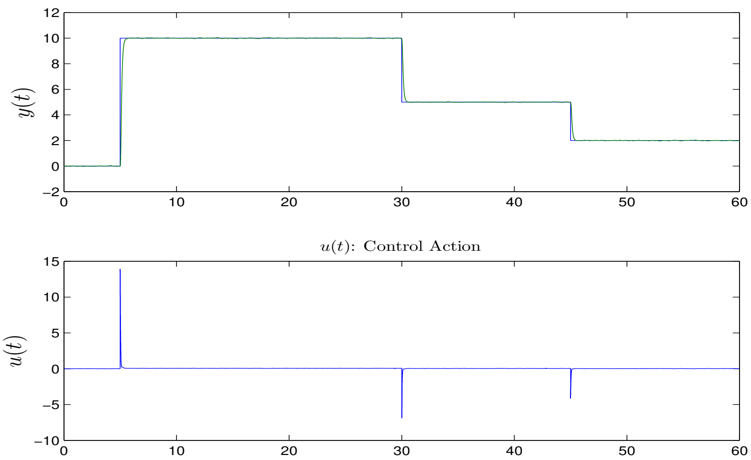


Figure 10: Mixed sensitivity result - W_p, Wu, Wt design - multistep command

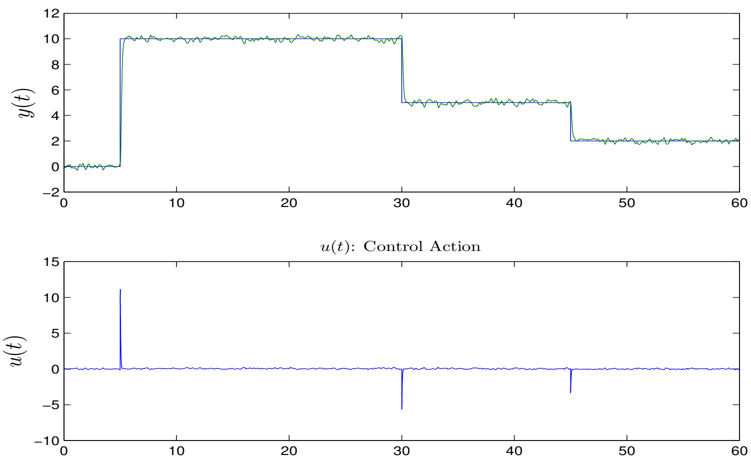


Figure 11: Mixed sensitivity result - W_p, Wu design - multistep command

9 Guaranteed Cost Control (GCC) of Uncertain Discrete Time Systems with State and Input Delays

Presentation is primarily based on [6] i.e. the paper by L. Yu and F. Gao "Optimal guaranteed cost control of discrete-time, uncertain systems with both state and input delays", Journal of the Franklin Institute, vol. 338, 2001, p.101–110.

The Generic Case and Three Special (Sub)Cases of the GCC Approach

- The most Generic Case1: GCC Synthesis for uncertain DT system with state and input delays i.e. $x_{k+1} = (A + \Delta A)x_k + (A_1 + \Delta A_1)x_{k-d} + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$
- Case2: GCC Synthesis for uncertain DT systems with only Input Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$
- Case3: GCC Synthesis for uncertain DT systems with only State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (A_1 + \Delta A_1)x_{k-d}$
- Case4: GCC Synthesis for uncertain DT systems without Input or State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k$

9.1 The generic GCC Problem Setup & closed–loop Stability Analysis

Open–loop DT system with state and input delays and uncertain dynamics

$$x_{k+1} = (A + \Delta A)x_k + (A_1 + \Delta A_1)x_{k-d} + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h} \tag{83}$$

with $x \in \Re^n$ and $u \in \Re^m$

- d and h are unknown constant integers representing the number of delay units in the state and input, respectively, bounded as $0 \le d \le d^*$, $0 \le h \le h^*$ with bounds $d^*,\ h^*$ being known
- $A,\ A_1,\ B,\ B_1$ are known real constant matrices of appropriate dimensions
- uncertain matrices $\Delta A,\ \Delta B,\ \Delta A_1,\ \Delta B_1$ represent time-varying parameter uncertainties in the system model, satisfying

$$[\Delta A\ \Delta B\ \Delta A_1\ \Delta B_1] = DF\,[E_a\ E_b\ E_d\ E_h] \tag{84}$$

- $D,\ E_a\ E_b\ E_d\ E_h$ are known real constant matrices of appropriate dimensions describing the structure of uncertainties
- the unknown (time-varying) matrix F satisfies $F^T F \le I,\ \ \forall k$

Associated with the uncertain open–loop system (83) is the cost function

$$J = \sum_{k=0}^{\infty} \left[x_k^T Q x_k + u_k^T R u_k\right] \tag{85}$$

with $Q,\ R > 0$ being symmetric and positive definite matrices of appropriate dimensions. Assuming that the system state is available for feedback, the objective is the design of a memoryless state feedback control law $u_k^* = Kx_k$, such that for any admissible uncertainty F the resulting closed–loop system is not only asymptotically stable but also guarantees the satisfaction of the bound $J \le J^*$

with the constant positive scalar J^* being independent of the uncertainties. Then J^* is said to be a guaranteed cost and u_k^* is said to be a guaranteed cost control law. The rationale behind this last objective is to incorporate performance objectives into the (stabilization) design procedure.

Closing the loop in (83) with $u_k = Kx_k$, the closed–loop dynamics are

$$\begin{aligned} x_{k+1} &= [A + BK + \Delta A + \Delta BK]\, x_k + [B_1 + \Delta B_1]\, Kx_{k-h} + \\ &\quad [A_1 + \Delta A_1]\, x_{k-d} \\ &= [A + BK + DF(E_a + E_b K)]\, x_k + [B_1 + DFE_h]\, Kx_{k-h} + \\ &\quad [A_1 + DFE_d]\, x_{k-d} \\ &\triangleq A_C(k)x_k + B_H(k)Kx_{k-h} + A_D(k)x_{k-d} \end{aligned} \tag{86}$$

with the uncertain matrices $A_C,\ B_H,\ A_D$, defined as

$$A_C \triangleq A + BK + DF(E_a + E_b K), \quad B_H \triangleq B_1 + DFE_h, \quad A_D \triangleq A_1 + DFE_d \tag{87}$$

The cost function associated with the closed–loop system (86) is

$$J_{cl} = \sum_{k=0}^{\infty} x_k^T \left[Q + K^T RK\right] x_k \tag{88}$$

9.2 GCC Analysis: Sufficient condition for the existence of SSF solution to GCC

The symbol "$*$" induces symmetry as usual in the LMI literature.

Theorem 40 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller if there exist symmetric positive definite matrices $P,\ P_d \in \Re^{n\times n}, T \in \Re^{m\times m}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^T PA_D & A_C^T PB_H \\ * & A_D^T PA_D - P_d & A_D^T PB_H \\ * & * & B_H^T PB_H - T \end{bmatrix} < 0 \tag{89}$$

where

$$\begin{aligned} \Pi &\triangleq A_C^T PA_C \underbrace{-P + P_d + K^T TK + Q + K^T RK} \\ &\triangleq A_C^T PA_C + \Lambda \end{aligned} \tag{90}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices $A_C,\ B_H,\ A_D$ already defined in (87). Moreover the closed-loop cost function satisfies

$$\begin{aligned} J_{cl} \leq J^* &\triangleq x_0^T Px_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} + \sum_{i=1}^{h} x_{-i}^T K^T TKx_{-i} \\ &\leq \lambda_{max}(U^T PU) + d^* \lambda_{max}(U^T P_d U) + h^* \lambda_{max}(U^T K^T TKU) \end{aligned} \tag{91}$$

Proof: Defining

- the "positive with respect to x_k" function

$$V_k^{tot} = V_k^1 + V_k^2 + V_k^3 = x_k^T P x_k + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i} + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j} \tag{92}$$

  with $P,\ P_d,\ T > 0$ being symmetric positive definite matrices of appropriate dimensions,

- the augmented state vector $\xi_k \triangleq \begin{bmatrix} x_k \\ x_{k-d} \\ K x_{k-h} \end{bmatrix} \in \Re^{(2n+m)}$ which allows to write the closed-loop dynamics (86) as
  $x_{k+1} = A_C x_k + A_D x_{k-d} + B_H K x_{k-h} = [A_C \ \ A_D \ \ B_H]\,\xi_k$,

the forward difference $\Delta V_k = V_{k+1}^{tot} - V_k^{tot}$ along the trajectories of the closed-loop system (86) can be expressed in terms of ξ_k as follows:

ΔV_k^3–term:

$$\begin{aligned} \Delta V_k^3 &= \left[\sum_{j=1}^{h} x_{k+1-j}^T K^T T K x_{k+1-j}\right] - \left[\sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j}\right] \\ &= x_k^T K^T T K x_k - x_{k-h}^T K^T T K x_{k-h} \\ &= \xi_k^T \begin{pmatrix} K^T T K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -T \end{pmatrix} \xi_k \end{aligned} \tag{93}$$

ΔV_k^2–term:

$$\begin{aligned} \Delta V_k^2 &= \left[\sum_{i=1}^{d} x_{k+1-i}^T P_d x_{k+1-i}\right] - \left[\sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}\right] = x_k^T P_d x_k - x_{k-d}^T P_d x_{k-d} \\ &= \xi_k^T \begin{pmatrix} P_d & 0 & 0 \\ 0 & -P_d & 0 \\ 0 & 0 & 0 \end{pmatrix} \xi_k \end{aligned} \tag{94}$$

ΔV_k^1–term:

$V_k^1 = x_k^T P x_k$ can be written as $\xi_k^T \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xi_k$ with the matrix dimension being $(2n+m) \times (2n+m)$. Using $x_{k+1} = [A_C \ \ A_D \ \ B_H]\,\xi_k$ the V_{k+1}^1 term writes as

$$\begin{aligned} V_{k+1}^1 &= x_{k+1}^T P x_{k+1} = \xi_k^T \,[A_C \ \ A_D \ \ B_H]^T\, P\, [A_C \ \ A_D \ \ B_H]\,\xi_k \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D & A_D^T P B_H \\ * & * & B_H^T P B_H \end{bmatrix} \xi_k \end{aligned}$$

and hence

$$\Delta V_k^1 = V_{k+1}^1 - V_k^1 = \xi_k^T \begin{bmatrix} A_C^T P A_C - P & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D & A_D^T P B_H \\ * & * & B_H^T P B_H \end{bmatrix} \xi_k \tag{95}$$

Combining (93),(94),(95)

$$\begin{aligned}\Delta V_k^{tot} &= \Delta V_k^1 + \Delta V_k^2 + \Delta V_k^3 \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C - P + P_d + K^T T K & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D - P_d & A_D^T P B_H \\ * & * & B_H^T P B_H - T \end{bmatrix} \xi_k\end{aligned}$$

Recalling the definition $\Pi \triangleq A_C^T P A_C - P + P_d + K^T T K + Q + K^T R K$ from (187), the (1, 1)-element writes as $\Pi - (Q + K^T R K)$ and the previous expression for ΔV_k^{tot} becomes

$$\begin{aligned}\Delta V_k^{tot} = \xi_k^T &\begin{bmatrix} \Pi - (Q + K^T R K) & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D - P_d & A_D^T P B_H \\ * & * & B_H^T P B_H - T \end{bmatrix} \xi_k \\ = \xi_k^T &\left(\begin{bmatrix} \Pi & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D - P_d & A_D^T P B_H \\ * & * & B_H^T P B_H - T \end{bmatrix} - \begin{bmatrix} (Q + K^T R K) & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \right) \xi_k\end{aligned}$$

From the assumption about the negative definiteness of the matrix appearing in Theorem 68, it is clear that the "wish" for $\Delta V_k^{tot} < 0$ is indeed satisfied since (in that case)

$$\begin{aligned}\Delta V_k^{tot} &< -\xi_k^T \begin{bmatrix} Q + K^T R K & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \xi_k \\ &= -x_k^T \left[Q + K^T R K \right] x_k \\ &\leq -\lambda_{\min}(Q + K^T R K) \|x_k\|^2 < 0. \qquad (96)\end{aligned}$$

Noting that $Q + K^T R K > 0$, the fact that $\Delta V_k^{tot} < 0$ implies asymptotic (quadratic) stability (must use "partial" stability arguments). Furthermore

$$x_k^T \left[Q + K^T R K \right] x_k \leq -\Delta V_k^{tot} = V_k^{tot} - V_{k+1}^{tot}$$

Summing both sides of the last inequality from $k = 0$ to $k = N$ yields $\sum_{k=0}^{N} x_k^T \left[Q + K^T R K \right] x_k \leq V_0^{tot} - V_N^{tot}$.

Letting $N \to \infty$, while using the already proven (asymptotic) stability of the closed-loop system (i.e. that as $N \to \infty$ both $x_N \to 0$ and $V_N^{tot}(x_N) \to 0$),

$$J_{cl} = \sum_{k=0}^{\infty} x_k^T \left[Q + K^T R K \right] x_k \leq V_0^{tot}$$

$$= x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \triangleq J^* \qquad (97)$$

The guaranteed cost J^* in (97) depends only on the initial conditions, and not on the uncertainties.

To remove this dependence on the initial condition, there are two approaches: the stochastic approach and the deterministic method (adopted here) where it is assumed that the initial state of the system (83) is arbitrary but belongs to the set $x_{-i} \in \Re^n : x_{-i} = U u_i,\ u_i^T u_i \leq 1,\ i = 0, 1, 2, \ldots, \hat{d}$

where U is a given matrix and $\hat{d} = \max\{h^*,\ d^*\}$. The cost bound then obeys

$$\begin{aligned} J^* &\triangleq x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \\ &\leq \lambda_{max}(U^T P U) + d^* \lambda_{max}(U^T P_d U) + h^* \lambda_{max}(U^T K^T T K U) \end{aligned} \tag{98}$$

This completes the proof of the theorem. □

9.3 GCC Synthesis for systems with state and input delay

Theorem 41 *For the uncertain system (83) and the cost function (85) there exist symmetric positive-definite matrices P, T such that matrix inequality (89) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}, M = P_d^{-1} \in \Re^{n\times n}$, $N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & A_1M & B_1N & 0 & 0 & 0 & 0 & 0 \\ * & -S & 0 & 0 & (E_aS+E_bW)^T & S^T & W^T & S & W^T \\ * & * & -M & 0 & ME_d^T & 0 & 0 & 0 & 0 \\ * & * & * & -N & NE_h^T & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -M & 0 & 0 & 0 \\ * & * & * & * & * & 0 & -N & 0 & 0 \\ * & * & * & * & * & 0 & 0 & -Q^{-1} & 0 \\ * & * & * & * & * & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0. \tag{99}$$

Furthermore, if matrix inequality (189) has a feasible solution in terms of the variables $\{\epsilon, W, S\}$ then the state feedback control law $u_k = WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies

$$J \le (1+h^*)\lambda_{\max}(U^TS^{-1}U) + d^*\lambda_{\max}(U^TM^{-1}U) \tag{100}$$

Proof: Recalling from (187) the definitions
$\Pi \triangleq A_C^TPA_C - P + P_d + K^TTK + Q + K^TRK \triangleq A_C^TPA_C + \Lambda$, with
$\Lambda \triangleq -P + P_d + K^TTK + Q + K^TRK = \Lambda^T$,
the matrix inequality (89) in Theorem 68 i.e.

$$\begin{bmatrix} \Pi & A_C^TPA_D & A_C^TPB_H \\ * & A_D^TPA_D - P_d & A_D^TPB_H \\ * & * & B_H^TPB_H - T \end{bmatrix} < 0$$

can be decomposed as

$$\begin{bmatrix} A_C^TPA_C & A_C^TPA_D & A_C^TPB_H \\ * & A_D^TPA_D & A_D^TPB_H \\ * & * & B_H^TPB_H \end{bmatrix} + \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & -P_d & 0 \\ 0 & 0 & -T \end{bmatrix} =$$

$$\begin{bmatrix} A_C^T \\ A_D^T \\ B_H^T \end{bmatrix} P\,[A_C \;\; A_D \;\; B_H] + \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & -P_d & 0 \\ 0 & 0 & -T \end{bmatrix} < 0 \tag{101}$$

which by Schur complement (see Lemma 17) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_C & A_D & B_H \\ A_C^T & \Lambda & 0 & 0 \\ A_D^T & 0 & -P_d & 0 \\ B_H^T & 0 & 0 & -T \end{bmatrix} < 0 \tag{102}$$

Substituting in (102) the defining expressions of the uncertain matrices from (87), i.e.

- $A_C = A + BK + DF(E_a + E_bK)$
- $B_H = B_1 + DFE_h$
- $A_D = A_1 + DFE_d$

and separating the nominal from the uncertain parts (i.e. those including matrix F), (102) can be equivalently written as

$$\begin{bmatrix} -P^{-1} & A+BK & A_1 & B_1 \\ (A+BK)^T & \Lambda & 0 & 0 \\ A_1^T & 0 & -P_d & 0 \\ B_1^T & 0 & 0 & -T \end{bmatrix} +$$

$$\begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & (E_a+E_bK) & E_d & E_h \end{bmatrix} +$$

$$\begin{bmatrix} 0 & (E_a+E_bK) & E_d & E_h \end{bmatrix}^T F^T \begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0 \tag{103}$$

Inequality (193) is clearly of the form $G + M\Delta N + N^T\Delta^T M^T < 0$, with $\Delta \to F$, $M \to \begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}$ hence $\epsilon MM^T = \begin{bmatrix} +\epsilon DD^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $N \to \begin{bmatrix} 0 & (E_a+E_bK) & E_d & E_h \end{bmatrix}$ and its "G"–part symmetric.

Hence Lemma 5 can be used to transform (193) into the following equivalent "$G + \epsilon MM^T + \frac{1}{\epsilon}N^TRN$" inequality (valid $\forall$ admissible F and $\epsilon > 0$)

$$(193) \quad \Leftrightarrow \quad \exists\ \epsilon > 0, \begin{bmatrix} -P^{-1}+\epsilon DD^T & A+BK & A_1 & B_1 \\ (A+BK)^T & \Lambda & 0 & 0 \\ A_1^T & 0 & -P_d & 0 \\ B_1^T & 0 & 0 & -T \end{bmatrix}$$

$$+ \frac{1}{\epsilon} \begin{bmatrix} 0 \\ (E_a+E_bK)^T \\ E_d^T \\ E_h^T \end{bmatrix} \begin{bmatrix} 0 & (E_a+E_bK) & E_d & E_h \end{bmatrix} < 0, \tag{104}$$

which, by Schur Complement is equivalent ($\Leftrightarrow$) to

$$\begin{bmatrix} -P^{-1}+\epsilon DD^T & A+BK & A_1 & B_1 & 0 \\ (A+BK)^T & \Lambda & 0 & 0 & (E_a+E_bK)^T \\ A_1^T & 0 & -P_d & 0 & E_d^T \\ B_1^T & 0 & 0 & -T & E_h^T \\ 0 & (E_a+E_bK) & E_d & E_h & -\epsilon I \end{bmatrix} < 0 \tag{105}$$

Remark 42 (the important trick here is to leave the term

$\frac{1}{\epsilon}\begin{bmatrix} 0 \\ (E_a+E_bK)^T \\ E_d^T \\ E_h^T \end{bmatrix}\begin{bmatrix} 0 & (E_a+E_bK) & E_d & E_h \end{bmatrix}$ in the LMI (194) "as is" – do not multiply)

Introducing the variables

$$S = P^{-1}, \quad M = P_d^{-1}, \quad N = T^{-1}, \quad W = KP^{-1} = KS \tag{106}$$

and performing a congruent transformation on inequality (194) by pre- and post- multiplying both sides of it by the nonsingular, symmetric block–diagonal matrix $\mathrm{diag}(I, P^{-1}, P_d^{-1}, T^{-1}, I) = \mathrm{diag}(I, S, M, N, I)$, this last LMI is equivalently transformed into:

$$\text{LMI } \{[194]\} < 0 \Leftrightarrow \ \mathrm{diag}(I,S,M,N,I) \text{ LMI } \{[194]\} \ \mathrm{diag}(I,S,M,N,I) < 0$$

pre- multiplying:

$$\mathrm{diag}(I,S,M,N,I)\begin{bmatrix} -S+\epsilon DD^T & A+BK & A_1 & B_1 & 0 \\ (A+BK)^T & \Lambda & 0 & 0 & (E_a+E_bK)^T \\ A_1^T & 0 & -P_d & 0 & E_d^T \\ B_1^T & 0 & 0 & -T & E_h^T \\ 0 & (E_a+E_bK) & E_d & E_h & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S+\epsilon DD^T & A+BK & A_1 & B_1 & 0 \\ S(A+BK)^T & S\Lambda & 0 & 0 & S(E_a+E_bK)^T \\ MA_1^T & 0 & -I & 0 & ME_d^T \\ NB_1^T & 0 & 0 & -I & NE_h^T \\ 0 & (E_a+E_bK) & E_d & E_h & -\epsilon I \end{bmatrix}$$

post- multiplying:

$$\begin{bmatrix} -S+\epsilon DD^T & A+BK & A_1 & B_1 & 0 \\ S(A+BK)^T & S\Lambda & 0 & 0 & S(E_a+E_bK)^T \\ MA_1^T & 0 & -I & 0 & ME_d^T \\ NB_1^T & 0 & 0 & -I & NE_h^T \\ 0 & (E_a+E_bK) & E_d & E_h & -\epsilon I \end{bmatrix}\mathrm{diag}(I,S,M,N,I) =$$

$$\begin{bmatrix} -S+\epsilon DD^T & (A+BK)S & A_1M & B_1N & 0 \\ S(A+BK)^T & S\Lambda S & 0 & 0 & S(E_a+E_bK)^T \\ MA_1^T & 0 & -M & 0 & ME_d^T \\ NB_1^T & 0 & 0 & -N & NE_h^T \\ 0 & (E_a+E_bK)S & E_dM & E_hN & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S+\epsilon DD^T & (AS+BW) & A_1M & B_1N & 0 \\ (AS+BW)^T & S\Lambda S & 0 & 0 & (E_aS+E_bW)^T \\ MA_1^T & 0 & -M & 0 & ME_d^T \\ NB_1^T & 0 & 0 & -N & NE_h^T \\ 0 & (E_aS+E_bW) & E_dM & E_hN & -\epsilon I \end{bmatrix} < 0 \tag{107}$$

Using now

- the fact that $S = S^T$,
- the definition of Λ from(187),
- the definitions in (106),

the (2, 2)–element $S\Lambda S$ in the last inequality (195) writes as

$$\begin{aligned} S\Lambda S &= S[-P + P_d + K^T TK + Q + K^T RK]S \\ &= S[-S^{-1} + M^{-1} + K^T N^{-1} K + Q + K^T RK]S \\ &= -S + SM^{-1}S + W^T N^{-1} W + SQS + W^T RW. \end{aligned}$$

Now

- single out the $-S$ term,
- and express the (remaining) term
$SM^{-1}S + W^T N^{-1} W + SQS + W^T RW$ as
$\tilde{M}^T \mathrm{diag}(M^{-1},\ N^{-1},\ Q,\ R)\tilde{M}$ with $\tilde{M} \triangleq \begin{bmatrix} 0 & S & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 \\ 0 & S & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 \end{bmatrix}$.

(Verify by carrying out the matrix multiplications. The result is a 5×5 "all–zero–matrix" except its (2, 2)–element which is $SM^{-1}S + W^T N^{-1} W + SQS + W^T RW$)

Then LMI (195) becomes

$$\begin{bmatrix} -S + \epsilon DD^T & (AS + BW) & A_1 M & B_1 N & 0 \\ (AS + BW)^T & -S & 0 & 0 & (E_a S + E_b W)^T \\ MA_1^T & 0 & -M & 0 & ME_d^T \\ NB_1^T & 0 & 0 & -N & NE_h^T \\ 0 & (E_a S + E_b W) & E_d M & E_h N & -\epsilon I \end{bmatrix} +$$
$$\tilde{M}^T \begin{bmatrix} M^{-1} & 0 & 0 & 0 \\ 0 & N^{-1} & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & R \end{bmatrix} \tilde{M} < 0 \tag{108}$$

which by Schur complement (see Remark 43 below) is equivalent to the LMI (189) of Theorem 69. This completes the proof of the theorem. $\square$.

(Proof of (100) still missing !!!)

Remark 43 *Use the fact that*

$$\begin{bmatrix} M^{-1} & 0 & 0 & 0 \\ 0 & N^{-1} & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & R \end{bmatrix} = -\left\{ \begin{bmatrix} -M & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \\ 0 & 0 & -Q^{-1} & 0 \\ 0 & 0 & 0 & -R^{-1} \end{bmatrix} \right\}^{-1}$$

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 69, it is easy to prove the following Corollary.

Corollary 44 *For the uncertain system (83) (with input and state delays) there exist symmetric positive-definite matrices* P, P_d, T *such that* $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ *(with* $V_k^{tot} = x_k^T P x_k + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j} + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}$ *defined in (191)) holds for all admissible uncertainties if and only if there exist a positive scalar* $\epsilon > 0$*, a matrix* $W \in \Re^{m\times n}$ *and symmetric positive definite matrix* $S = P^{-1} \in \Re^{n\times n}$*,* $M = P_d^{-1} \in \Re^{m\times m}$*,* $N = T^{-1} \in \Re^{m\times m}$ *such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & A_1M & B_1N & 0 & 0 & 0 \\ * & -S & 0 & 0 & (E_aS+E_bW)^T & S & W^T \\ * & * & -M & 0 & ME_d^T & 0 & 0 \\ * & * & * & -N & NE_h^T & 0 & 0 \\ * & * & * & * & -\epsilon I & 0 & 0 \\ * & * & * & * & * & -M & 0 \\ * & * & * & * & * & 0 & -N \end{bmatrix} < 0. \quad (109)$$

Furthermore, if matrix inequality (190) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$*, then the state feedback control law* $u_k = WS^{-1}x_k = Kx_k$ *is a robustly stabilizing control law.* □

LMIs (190) is a "subset of the Generic" LMI (189) formally derived after removing the "appropriate" rows and columns i.e. the last two rows and columns containing the matrices Q, R.

10 Three interesting (Sub)Cases of the generic GCC Problem (& Application to NCS)

- Case1: GCC Synthesis for uncertain DT systems with only Input Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$
- SubCase1a: Application of Case1 to NCS with "small" input delay
- Case2: GCC Synthesis for uncertain DT systems with only State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (A_1 + \Delta A_1)x_{k-d}$
- Case3: GCC Synthesis for uncertain DT systems without Input or State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k$

Note that –as expected– for all the above cases the synthesis LMIs (to be derived) are a "subset of the Generic" LMI (189) after deleting the "appropriate" rows and columns.

10.1 GCC Analysis & Synthesis for uncertain DT systems with (only) Input Delay

Open–loop DT system with state and input delay (NO STATE DELAY $\Rightarrow A_1 = \Delta A_1 = E_d = 0$) and uncertain dynamics

$$x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h} \tag{110}$$

with $x \in \Re^n$ and $u \in \Re^m$ and (since $A_1 = \Delta A_1 = E_d = 0$)

$$[\Delta A \; \Delta B \; \Delta B_1] = DF \, [E_a \; E_b \; E_h] \tag{111}$$

with unknown (time-varying) matrix F satisfying $F^T F \leq I$. Furthermore h is an unknown constant integer (delay units in the input), bounded as $0 \leq h \leq h^*$ with h^* known.

The cost function is the same as in (85). The closed–loop dynamics with $u_k = Kx_k$ are

$$\begin{aligned} x_{k+1} &= [A + BK + \Delta A + \Delta BK] \, x_k + [B_1 + \Delta B_1] \, Kx_{k-h} \\ &= [A + BK + DF(E_a + E_b K)] \, x_k + [B_1 + DFE_h] \, Kx_{k-h} \\ &\triangleq A_C(k)x_k + B_H(k)Kx_{k-h} \end{aligned} \tag{112}$$

with the uncertain matrices A_C, B_H, defined as

$$A_C \triangleq A + BK + DF(E_a + E_b K), \quad B_H \triangleq B_1 + DFE_h \tag{113}$$

10.1.1 GCC Analysis for systems with (only) Input Delay

The sufficient condition for the existence of memoryless state feedback GCC law is a special "case" of Theorem (68)

Theorem 45 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller for (110) if there exist symmetric positive definite matrices $P \in \Re^{n\times n}, T \in \Re^{m\times m}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^T P B_H \\ * & B_H^T P B_H - T \end{bmatrix} < 0 \tag{114}$$

where

$$\begin{aligned}\Pi &\triangleq A_C^T P A_C \underbrace{-P + K^T T K + Q + K^T R K} \\ &\triangleq A_C^T P A_C + \Lambda \end{aligned} \tag{115}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices A_C, B_H already defined in (113). Moreover the closed-loop cost function satisfies

$$J_{cl} \leq J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \leq \lambda_{max}(U^T P U) + h^* \lambda_{max}(U^T K^T T K U) \tag{116}$$

Proof: Defining

- the "positive with respect to x_k" scalar function

$$V_k^{tot} = V_k^1 + V_k^3 = x_k^T P x_k + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j} \tag{117}$$

  with $P,\ T > 0$ being SPDef matrices of appropriate dimensions,

- the augmented state vector $\xi_k \triangleq \begin{bmatrix} x_k \\ K x_{k-h} \end{bmatrix} \in \Re^{(n+m)}$ which allows to write the closed-loop dynamics (112) as $x_{k+1} = A_C(k) x_k + B_H(k) K x_{k-h} = [A_C \ \ B_H]\,\xi_k$

the forward difference $\Delta V_k = V_{k+1}^{tot} - V_k^{tot}$ along the trajectories of the closed-loop system (112) can be expressed in terms of ξ_k as follows:

ΔV_k^1–term:

$V_k^1 = x_k^T P x_k$ can be written as $\xi_k^T \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \xi_k$

Using $x_{k+1} = [A_C \ \ B_H]\,\xi_k$, the V_{k+1}^1 term writes as

$$\begin{aligned} V_{k+1}^1 &= x_{k+1}^T P x_{k+1} = \xi_k^T [A_C \ \ B_H]^T P [A_C \ \ B_H] \xi_k \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C & A_C^T P B_H \\ * & B_H^T P B_H \end{bmatrix} \xi_k \end{aligned}$$

and hence

$$\Delta V_k^1 = V_{k+1}^1 - V_k^1 = \xi_k^T \begin{bmatrix} A_C^T P A_C - P & A_C^T P B_H \\ * & B_H^T P B_H \end{bmatrix} \xi_k \tag{118}$$

ΔV_k^3–term:

$$\begin{aligned} \Delta V_k^3 &= \left[\sum_{j=1}^{h} x_{k+1-j}^T K^T T K x_{k+1-j} \right] - \left[\sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j} \right] \\ &= x_k^T K^T T K x_k - x_{k-h}^T K^T T K x_{k-h} \\ &= \xi_k^T \begin{pmatrix} K^T T K & 0 \\ 0 & -T \end{pmatrix} \xi_k \end{aligned} \tag{119}$$

Combining (119),(118)

$$\begin{aligned} \Delta V_k^{tot} &= \Delta V_k^1 + \Delta V_k^3 \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C - P + K^T T K & A_C^T P B_H \\ * & B_H^T P B_H - T \end{bmatrix} \xi_k \end{aligned}$$

Recalling the definition $\Pi \triangleq A_C^T P A_C - P + K^T T K + Q + K^T R K \triangleq A_C^T P A_C + \Lambda$ (see (115)) the $(1,1)$-element writes as $\Pi - (Q + K^T R K)$ and the previous expression for ΔV_k^{tot} becomes

$$\Delta V_k^{tot} = \xi_k^T \left(\begin{bmatrix} \Pi & A_C^T P B_H \\ * & B_H^T P B_H - T \end{bmatrix} - \begin{bmatrix} (Q + K^T R K) & 0 \\ * & 0 \end{bmatrix} \right) \xi_k$$

From the assumption about the negative definiteness of the matrix appearing in Theorem 45, it is clear that the "wish" for $\Delta V_k^{tot} < 0$ is indeed satisfied since (in that case)

$$\begin{aligned} \Delta V_k^{tot} &< -\xi_k^T \begin{bmatrix} (Q + K^T RK) & 0 \\ * & 0 \end{bmatrix} \xi_k \\ &= -x_k^T \left[Q + K^T RK \right] x_k \\ &\leq -\lambda_{\min}(Q + K^T RK)\|x_k\|^2 < 0. \end{aligned} \tag{120}$$

Noting that $Q + K^T RK > 0$ the last inequality implies asymptotic (quadratic) stability (must use "partial" stability arguments).

Following the same arguments as in Theorem (68) the closed-loop cost function satisfies

$$\begin{aligned} J_{cl} &= \sum_{k=0}^{\infty} x_k^T \left[Q + K^T RK \right] x_k \leq V_0^{tot} \\ &= x_0^T P x_0 + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \triangleq J^* \end{aligned} \tag{121}$$

The guaranteed cost J^* in (121) depends only on the initial conditions, and not on the uncertainties.

Adopting the deterministic method to remove this dependence on the initial condition, it is assumed that the initial state of the system (110) is arbitrary but belongs to the set $x_{-i} \in \Re^n : x_{-i} = U u_i,\ u_i^T u_i \leq 1,\ i = 0, 1, 2, \ldots, h^*$ where U is a given matrix. The cost bound then obeys

$$J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \leq \lambda_{max}(U^T PU) + h^* \lambda_{max}(U^T K^T T K U) \tag{122}$$

This completes the proof of the theorem. □

10.1.2 GCC Synthesis for systems with (only) Input Delay

Theorem 46 *For the uncertain (input delayed) system (110) and the cost function (85) there exist symmetric positive-definite matrices P, T such that matrix inequality (114) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & B_1 N & 0 & 0 & 0 & 0 \\ * & -S & 0 & (E_a S + E_b W)^T & W^T & S & W^T \\ * & * & -N & N E_h^T & 0 & 0 & 0 \\ * & * & * & -\epsilon I & 0 & 0 & 0 \\ * & * & * & * & -N & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & * & -R^{-1} \end{bmatrix} < 0. \tag{123}$$

Furthermore, if matrix inequality (123) has a feasible solution in terms of the variables {ϵ, W, S, N } then the state feedback control law $u_k = W S^{-1} x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies $J \leq (1 + h^*)\lambda_{\max}(U S^{-1} U)$

Remark 47 *Note that the LMI (123) of Theorem 46, results from the "Generic" LMI (189) after removing the third row/column (containing M, A_1, E_d which are "zero" matrices since they involve state-delay) and the sixth row/column (involve M, S) with M being a "zero" while matrix S is already constrained via LMI (123)*

Proof: Starting from the matrix inequality (114) in Theorem 45 i.e.

$$\begin{bmatrix} \Pi & A_C^T P B_H \\ * & B_H^T P B_H - T \end{bmatrix} < 0$$

and recalling from (115) the definitions

$$\Pi \triangleq A_C^T P A_C - P + K^T T K + Q + K^T R K \triangleq A_C^T P A_C + \Lambda$$

$$\Lambda \triangleq -P + K^T T K + Q + K^T R K = \Lambda^T$$

can decompose the matrix inequality (114) as

$$\begin{bmatrix} A_C^T P A_C & A_C^T P B_H \\ * & B_H^T P B_H \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & -T \end{bmatrix} =$$

$$\begin{bmatrix} A_C^T \\ B_H^T \end{bmatrix} P \begin{bmatrix} A_C & B_H \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & -T \end{bmatrix} < 0 \tag{124}$$

which by Schur complement (see Lemma 17) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_C & B_H \\ A_C^T & \Lambda & 0 \\ B_H^T & 0 & -T \end{bmatrix} < 0 \tag{125}$$

Substituting in (125) the defining expressions of the uncertain matrices from (113), i.e. $A_C = A + BK + DF(E_a + E_b K)$, $B_H = B_1 + DFE_h$...

...and separating the nominal from the uncertain parts (i.e. those including matrix F),

(125) can be equivalently written as

$$\begin{bmatrix} -P^{-1} & A + BK & B_1 \\ (A+BK)^T & \Lambda & 0 \\ B_1^T & 0 & -T \end{bmatrix} +$$

$$\begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & (E_a + E_b K) & E_h \end{bmatrix} +$$

$$\begin{bmatrix} 0 & (E_a + E_b K) & E_h \end{bmatrix}^T F^T \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}^T < 0 \tag{126}$$

Inequality (126) is clearly of the form $G + M\Delta N + N^T \Delta^T M^T < 0$, with $\Delta \to F$, $M \to \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$ hence $\epsilon M M^T = \begin{bmatrix} +\epsilon D D^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N \to \begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix}$ and its "G"–part symmetric.

Hence Lemma 5 can be used to transform (126) into the following equivalent "$G + \epsilon M M^T + \frac{1}{\epsilon} N^T R N$" matrix inequality (valid $\forall$ admissible uncertainties F and $\forall\ \epsilon > 0$)

$$(126) \quad \Leftrightarrow \quad \exists\ \epsilon > 0, \begin{bmatrix} -P^{-1} + \epsilon D D^T & A + BK & B_1 \\ (A+BK)^T & \Lambda & 0 \\ B_1^T & 0 & -T \end{bmatrix}$$

$$+ \quad \frac{1}{\epsilon} \begin{bmatrix} 0 \\ (E_a + E_b K)^T \\ E_d^T \end{bmatrix} \begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix} < 0,$$

which, by Schur Complement is equivalent ($\Leftrightarrow$) to

$$\begin{bmatrix} -P^{-1} + \epsilon D D^T & A + BK & B_1 & 0 \\ (A+BK)^T & \Lambda & 0 & (E_a + E_b K)^T \\ B_1^T & 0 & -T & E_h^T \\ 0 & (E_a + E_b K) & E_h & -\epsilon I \end{bmatrix} < 0 \tag{127}$$

Remark 48 (the important trick here is to leave the term $\frac{1}{\epsilon} \begin{bmatrix} 0 \\ (E_a + E_b K)^T \\ E_d^T \end{bmatrix} \begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix}$ in the last LMI "as is" – do not multiply)

Introducing the variables

$$S = P^{-1}, \quad N = T^{-1}, \quad W = KP^{-1} = KS \tag{128}$$

and (performing a congruent transformation on inequality (127) by) pre- and post- multiplying both sides of it by the nonsingular, symmetric block–diagonal matrix $\text{diag}(I, P^{-1}, T^{-1}, I) = \text{diag}(I, S, N, I)$,

$$(127) \Leftrightarrow \text{diag}(I, S, N, I)\ (127)\ \text{diag}(I, S, N, I) < 0$$

pre- multiplying:

$$\text{diag}(I, S, N, I)\begin{bmatrix} -S + \epsilon DD^T & A + BK & B_1 & 0 \\ (A + BK)^T & \Lambda & 0 & (E_a + E_b K)^T \\ B_1^T & 0 & -T & E_h^T \\ 0 & (E_a + E_b K) & E_h & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S + \epsilon DD^T & A + BK & B_1 & 0 \\ S(A + BK)^T & S\Lambda & 0 & S(E_a + E_b K)^T \\ NB_1^T & 0 & -I & NE_h^T \\ 0 & (E_a + E_b K) & E_h & -\epsilon I \end{bmatrix}$$

post- multiplying:

$$\begin{bmatrix} -S + \epsilon DD^T & A + BK & B_1 & 0 \\ S(A + BK)^T & S\Lambda & 0 & S(E_a + E_b K)^T \\ NB_1^T & 0 & -I & NE_h^T \\ 0 & (E_a + E_b K) & E_h & -\epsilon I \end{bmatrix}\text{diag}(I, S, N, I) =$$

$$\begin{bmatrix} -S + \epsilon DD^T & (A + BK)S & B_1 N & 0 \\ S(A + BK)^T & S\Lambda S & 0 & S(E_a + E_b K)^T \\ NB_1^T & 0 & -N & NE_h^T \\ 0 & (E_a + E_b K)S & E_h N & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S + \epsilon DD^T & (AS + BW) & B_1 N & 0 \\ (AS + BW)^T & S\Lambda S & 0 & (E_a S + E_b W)^T \\ NB_1^T & 0 & -N & NE_h^T \\ 0 & (E_a S + E_b W) & E_h N & -\epsilon I \end{bmatrix} \tag{129}$$

Using

- the fact that $S = S^T$,
- the definition of Λ from(115),
- the definitions in (128),

the $(2, 2)$–element $S\Lambda S$ in (129) writes as

$$\begin{aligned} S\Lambda S &= S[-P + K^T TK + Q + K^T RK]S \\ &= S[-S^{-1} + K^T N^{-1} K + Q + K^T RK]S \\ &= -S + W^T N^{-1} W + SQS + W^T RW. \end{aligned}$$

Now

- single out the $-S$ term,
- and express the (remaining) term $W^T N^{-1} W + SQS + W^T RW$ as $\tilde{M}^T \text{diag}(N^{-1}, Q, R)\tilde{M}$ with $\tilde{M} \triangleq \begin{bmatrix} 0 & W & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & W & 0 & 0 \end{bmatrix}$

Then LMI (129) becomes

$$\begin{bmatrix} -S + \epsilon DD^T & (AS + BW) & B_1 N & 0 \\ (AS + BW)^T & -S & 0 & (E_a S + E_b W)^T \\ NB_1^T & 0 & -N & NE_h^T \\ 0 & (E_a S + E_b W) & E_h N & -\epsilon I \end{bmatrix} +$$

$$\tilde{M}^T \begin{bmatrix} N^{-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \tilde{M} < 0 \tag{130}$$

which by Schur complement (see Remark 49 below) is equivalent to the LMI (123) of Theorem 46. This completes the proof of the theorem. □

Remark 49 *Use* $\begin{bmatrix} N^{-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} = -\left\{ \begin{bmatrix} -N & 0 & 0 \\ 0 & -Q^{-1} & 0 \\ 0 & 0 & -R^{-1} \end{bmatrix} \right\}^{-1}$

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 46, it is easy to prove the following Corollary.

Corollary 50 *For the uncertain (input delayed) system (110) there exist symmetric positive-definite matrices P, T such that $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ (with V_k^{tot} defined as $V_k^{tot} = x_k^T P x_k + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j}$) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m \times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n \times n}$, $N = T^{-1} \in \Re^{m \times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & B_1 N & 0 & 0 \\ * & -S & 0 & (E_a S + E_b W)^T & W^T \\ * & * & -N & N E_h^T & 0 \\ * & * & * & -\epsilon I & 0 \\ * & * & * & * & 0 \\ * & * & * & * & -N \end{bmatrix} < 0. \tag{131}$$

Furthermore, if matrix inequality (131) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$, then the state feedback control law $u_k = WS^{-1}x_k = Kx_k$ is a robustly stabilizing control law. □

LMIs (131) is a "subset of the Generic" LMI (123) formally derived after removing the "appropriate" rows and columns i.e. the last two rows and columns containing the matrices Q, R.

10.2 GCC Analysis & Synthesis for uncertain DT systems with (only) State Delay

Open–loop DT system with (only) state delay and uncertain dynamics

$$x_{k+1} = (A + \Delta A)x_k + (A_1 + \Delta A_1)x_{k-d} + (B + \Delta B)u_k \tag{132}$$

with $x \in \Re^n$ and $u \in \Re^m$ and NO INPUT DELAY (hence $B_1 = \Delta B_1 = E_h = 0$)

$$[\Delta A \; \Delta A_1 \; \Delta B] = DF \, [E_a \; E_d \; E_b] \tag{133}$$

with the unknown (time-varying) matrix F satisfying $F^T F \leq I$. Furthermore d is an unknown constant integer (delay units in the state), bounded as $0 \leq d \leq d^*$ with d^* known. The closed–loop dynamics with $u_k = Kx_k$ are

$$\begin{aligned} x_{k+1} &= [A + BK + DF(E_a + E_b K)]\, x_k + [A_1 + DFE_d]\, x_{k-d} \\ &\triangleq A_C(k)x_k + A_D(k)x_{k-h} \end{aligned} \tag{134}$$

with the uncertain matrices A_C, A_D, defined as

$$A_C \triangleq A + BK + DF(E_a + E_b K), \;\; A_D \triangleq A_1 + DFE_d \tag{135}$$

10.2.1 GCC Analysis for systems with (only) State Delay

The sufficient condition for the existence of memoryless state feedback GCC law is a special "case" of Theorem (68)

Theorem 51 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller for (132) if there exist symmetric positive definite matrices P, $P_d \in \Re^{n\times n}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^T P A_D \\ * & A_D^T P A_D - P_d \end{bmatrix} < 0 \tag{136}$$

where

$$\begin{aligned} \Pi &\triangleq A_C^T P A_C \underbrace{-P + P_d + Q + K^T R K} \\ &\triangleq A_C^T P A_C + \Lambda \end{aligned} \tag{137}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices A_C, A_D already defined in (135). Moreover the closed-loop cost function satisfies

$$J_{cl} \leq J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} \leq \lambda_{max}(U^T P U) + d^* \lambda_{max}(U^T P_d U) \tag{138}$$

Proof 52 *Defining*

- *the "positive with respect to x_k" function*

$$V_k^{tot} = V_k^1 + V_k^2 = x_k^T P x_k + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i} \tag{139}$$

with P, $P_d > 0$ being SPDef matrices of appropriate dimensions $0 < P$, $P_d \in \Re^{n\times n}$,

- *the augmented state vector* $\xi_k \triangleq \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix} \in \Re^{2n}$ *which allows to write the closed-loop dynamics (134) as* $x_{k+1} = A_C(k)x_k + A_D(k)x_{k-h} = [A_C \;\; A_D]\,\xi_k$

the forward difference $\Delta V_k = V_{k+1}^{tot} - V_k^{tot}$ *along the trajectories of the closed-loop system (134) can be expressed in terms of* ξ_k *as follows:*

ΔV_k^1–term:

$V_k^1 = x_k^T P x_k$ *can be written as* $\xi_k^T \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \xi_k$

Using $x_{k+1} = [A_C \;\; A_D]\,\xi_k$, *the* V_{k+1}^1 *term writes as*

$$\begin{aligned} V_{k+1}^1 &= x_{k+1}^T P x_{k+1} = \xi_k^T [A_C \;\; A_D]^T P [A_C \;\; A_D]\,\xi_k \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C & A_C^T P A_D \\ * & A_D^T P A_D \end{bmatrix} \xi_k \end{aligned}$$

and

$$\Delta V_k^1 = V_{k+1}^1 - V_k^1 = \xi_k^T \begin{bmatrix} A_C^T P A_C - P & A_C^T P A_D \\ * & A_D^T P A_D \end{bmatrix} \xi_k \tag{140}$$

ΔV_k^2–term:

$$\begin{aligned} \Delta V_k^2 &= \left[\sum_{i=1}^{d} x_{k+1-i}^T P_d x_{k+1-i}\right] - \left[\sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}\right] \\ &= x_k^T P_d x_k - x_{k-d}^T P_d x_{k-d} \\ &= \xi_k^T \begin{pmatrix} P_d & 0 \\ 0 & -P_d \end{pmatrix} \xi_k \end{aligned} \tag{141}$$

Combining (141),(140)

$$\begin{aligned} \Delta V_k^{tot} &= \Delta V_k^1 + \Delta V_k^2 \\ &= \xi_k^T \begin{bmatrix} A_C^T P A_C - P + P_d & A_C^T P A_D \\ * & A_D^T P A_D - P_d \end{bmatrix} \xi_k \end{aligned}$$

Defining (see (137)) $\Pi \triangleq A_C^T P A_C - P + P_d + Q + K^T R K \triangleq A_C^T P A_C + \Lambda$ *the* $(1,1)$*-element writes as* $\Pi - (Q + K^T R K)$ *and the previous expression for* ΔV_k^{tot} *becomes*

$$\Delta V_k^{tot} = \xi_k^T \left(\begin{bmatrix} \Pi & A_C^T P A_D \\ * & A_D^T P A_D - P_d \end{bmatrix} - \begin{bmatrix} (Q + K^T R K) & 0 \\ * & 0 \end{bmatrix} \right) \xi_k$$

From the assumption about the negative definiteness of the matrix appearing in Theorem 51, it is clear that the "wish" for $\Delta V_k^{tot} < 0$ *is indeed satisfied since (in that case)*

$$\begin{aligned} \Delta V_k^{tot} &\leq -\xi_k^T \begin{bmatrix} (Q + K^T R K) & 0 \\ * & 0 \end{bmatrix} \xi_k \\ &= -x_k^T \left[Q + K^T R K\right] x_k \\ &\leq -\lambda_{\min}(Q + K^T R K)\|x_k\|^2 < 0. \end{aligned} \tag{142}$$

Noting that $Q + K^T R K > 0$, *the last inequality implies asymptotic (quadratic) stability (must use "partial" stability arguments).*

Following the same arguments as in Theorem (68) the closed-loop cost function

$$\begin{aligned} J_{cl} &= \sum_{k=0}^{\infty} x_k^T \left[Q + K^T R K\right] x_k \leq V_0^{tot} \\ &= x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} \triangleq J^* \end{aligned} \tag{143}$$

The guaranteed cost J^* *in (143) depends only on the initial conditions, and not on the uncertainties.*

Adopting the deterministic method to remove this dependence on the initial condition, it is assumed that the initial state of the system (132) is arbitrary but belongs to the set $x_{-i} \in \Re^n : x_{-i} = U u_i,\; u_i^T u_i \leq 1,\; i = 0, 1, 2, \ldots, d^*$ *where* U *is a given matrix. The cost bound then obeys*

$$J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} \leq \lambda_{max}(U^T P U) + d^* \lambda_{max}(U^T P_d U) \tag{144}$$

This completes the proof of the theorem.

10.2.2 GCC Synthesis for systems with (only) State Delay

Theorem 53 *For the uncertain (state delayed) system (132) and the cost function (85) there exist symmetric positive-definite matrices P, P_d such that matrix inequality (136) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $M = P_d^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & A_1M & 0 & 0 & 0 & 0 \\ (AS+BW)^T & -S & 0 & (E_aS+E_bW)^T & S & S & W^T \\ MA_1^T & 0 & -M & ME_d^T & 0 & 0 & 0 \\ 0 & (E_aS+E_bW) & E_dM & -\epsilon I & 0 & 0 & 0 \\ 0 & S & 0 & 0 & -M & 0 & 0 \\ 0 & S & 0 & 0 & 0 & -Q^{-1} & 0 \\ 0 & W & 0 & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0. \tag{145}$$

Furthermore, if matrix inequality (145) has a feasible solution in terms of the variables {ϵ, W, S, M } then the state feedback control law $u_k = WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies $J \le (d^*)\lambda_{\max}(US^{-1}U)$

Remark 54 *Note that the LMI (145) of Theorem 53, results from the "Generic" LMI (189) after removing the fourth row/column (involve B_1, N) and the seventh row/column (involve W, N).*

Proof 55 *Starting from the matrix inequality (136) in Theorem 51 i.e.*

$$\begin{bmatrix} \Pi & A_C^TPA_D \\ * & A_D^TPA_D - P_d \end{bmatrix} < 0$$

and recalling from (137) the definitions
$\Pi \triangleq A_C^TPA_C - P + P_d + Q + K^TRK \triangleq A_C^TPA_C + \Lambda$, *with*
$\Lambda \triangleq -P + P_d + Q + K^TRK = \Lambda^T$, *can decompose this LMI as*

$$\begin{bmatrix} A_C^TPA_C & A_C^TPA_D \\ * & A_D^TPA_D \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & -P_d \end{bmatrix} = \begin{bmatrix} A_C^T \\ A_D^T \end{bmatrix} P \, [A_C \;\; A_D] + \begin{bmatrix} \Lambda & 0 \\ 0 & -P_d \end{bmatrix} < 0 \tag{146}$$

which by Schur complement (see Lemma 17) is equivalent to

$$\begin{bmatrix} -P^{-1} & A_C & A_D \\ A_C^T & \Lambda & 0 \\ A_D^T & 0 & -P_d \end{bmatrix} < 0 \tag{147}$$

Substituting in (147) the defining expressions of the uncertain matrices from (135), i.e. $A_C = A + BK + DF(E_a + E_bK)$, $A_D \triangleq A_1 + DFE_d$, and separating the nominal from the uncertain parts (i.e. those including matrix F), (147) can be equivalently written as

$$\begin{bmatrix} -P^{-1} & A+BK & A_1 \\ (A+BK)^T & \Lambda & 0 \\ A_1^T & 0 & -P_d \end{bmatrix} + \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & (E_a+E_bK) & E_d \end{bmatrix} + \begin{bmatrix} 0 & (E_a+E_bK) & E_d \end{bmatrix}^T F^T \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}^T < 0 \tag{148}$$

Inequality (148) is clearly of the form $G + M\Delta N + N^T\Delta^T M^T < 0$, *with* $\Delta \to F$, $M \to \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$ *hence* $\epsilon M M^T = \begin{bmatrix} +\epsilon DD^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N \to \begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix}$ *and its "G"–part symmetric.*

Hence Lemma 5 can be used to transform (148) into the following equivalent "$G + \epsilon M M^T + \frac{1}{\epsilon} N^T R N$*" inequality (valid* $\forall$ *admissible uncertainties F and* $\forall\ \epsilon > 0$*)*

$$(148) \quad \Leftrightarrow \quad \exists\ \epsilon > 0, \begin{bmatrix} -P^{-1} + \epsilon DD^T & A + BK & A_1 \\ (A+BK)^T & \Lambda & 0 \\ A_1^T & 0 & -P_d \end{bmatrix} + \frac{1}{\epsilon}\begin{bmatrix} 0 \\ (E_a + E_b K)^T \\ E_d^T \end{bmatrix}\begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix} < 0 \tag{149}$$

which, by Schur Complement, is equivalent ($\Leftrightarrow$*) to*

$$\begin{bmatrix} -P^{-1} + \epsilon DD^T & A + BK & A_1 & 0 \\ (A+BK)^T & \Lambda & 0 & (E_a + E_b K)^T \\ A_1^T & 0 & -P_d & E_d^T \\ 0 & (E_a + E_b K) & E_d & -\epsilon I \end{bmatrix} < 0 \tag{150}$$

Remark 56 the important trick at this step is to leave the term $\frac{1}{\epsilon}\begin{bmatrix} 0 \\ (E_a + E_b K)^T \\ E_d^T \end{bmatrix}\begin{bmatrix} 0 & (E_a + E_b K) & E_d \end{bmatrix}$ in the last LMI "as is" – do not multiply

Introducing the variables

$$S = P^{-1}, \quad M = P_d^{-1}, \quad W = KP^{-1} = KS \tag{151}$$

and (performing a congruent transformation on inequality (150) by) pre- and post- multiplying both sides of it by the nonsingular, symmetric block–diagonal matrix $diag(I, P^{-1}, P_d^{-1}, I) = diag(I, S, M, I)$,

$$(150) \Leftrightarrow\ diag(I, S, M, I)\ (150)\ diag(I, S, M, I) < 0$$

post- multiplying:

$$\begin{bmatrix} -S + \epsilon DD^T & A + BK & A_1 & 0 \\ (A+BK)^T & \Lambda & 0 & (E_a + E_b K)^T \\ A_1^T & 0 & -P_d & E_d^T \\ 0 & (E_a + E_b K) & E_d & -\epsilon I \end{bmatrix} diag(I, S, M, I) =$$

$$\begin{bmatrix} -S + \epsilon DD^T & (A + BK)S & A_1 M & 0 \\ (A+BK)^T & \Lambda S & 0 & (E_a + E_b K)^T \\ A_1^T & 0 & -P_d M & E_d^T \\ 0 & (E_a + E_b K)S & E_d M & -\epsilon I \end{bmatrix}$$

and finally pre- multiplying:

$$diag(I, S, M, I)\begin{bmatrix} -S + \epsilon DD^T & (A + BK)S & A_1 M & 0 \\ (A+BK)^T & \Lambda S & 0 & (E_a + E_b K)^T \\ A_1^T & 0 & -I & E_d^T \\ 0 & (E_a + E_b K)S & E_d M & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S + \epsilon DD^T & (A + BK)S & A_1 M & 0 \\ S(A+BK)^T & S\Lambda S & 0 & S(E_a + E_b K)^T \\ MA_1^T & 0 & -M & ME_d^T \\ 0 & (E_a + E_b K)S & E_d M & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S + \epsilon DD^T & (AS + BW) & A_1 M & 0 \\ (AS + BW)^T & S\Lambda S & 0 & (E_a S + E_b W)^T \\ MA_1^T & 0 & -M & ME_d^T \\ 0 & (E_a S + E_b W) & E_d M & -\epsilon I \end{bmatrix} < 0 \tag{152}$$

Using

- *the fact that* $S = S^T$,
- *the definition of* Λ *from(137),*
- *the definitions in (151),*

the $(2,2)$*–element* $S\Lambda S$ *in (152) writes as*

$$\begin{aligned} S\Lambda S &= S[-P + P_d + Q + K^T RK]S \\ &= S[-S^{-1} + M^{-1} + Q + K^T RK]S \\ &= -S + S^T M^{-1} S + SQS + W^T RW. \end{aligned}$$

Now

- *single out the* $-S$ *term,*
- *and express the (remaining) term* $S^T M^{-1} S + SQS + W^T RW$ *as* $\tilde{M}^T diag(M^{-1},\ Q,\ R)\tilde{M}$ *with* $\tilde{M} \triangleq \begin{bmatrix} 0 & S & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & W & 0 & 0 \end{bmatrix}$,

(Verify by carrying out the matrix multiplications. The result is a 4×4 *"all–zero–matrix" except its* $(2,2)$*–element which is* $S^T M^{-1} S + SQS + W^T RW$*)*

LMI (152) becomes

$$\begin{bmatrix} -S + \epsilon DD^T & (AS + BW) & A_1 M & 0 \\ (AS + BW)^T & -S & 0 & (E_a S + E_b W)^T \\ MA_1^T & 0 & -M & ME_d^T \\ 0 & (E_a S + E_b W) & E_d M & -\epsilon I \end{bmatrix} + \tilde{M}^T \begin{bmatrix} M^{-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} \tilde{M} < 0 \tag{153}$$

which by Schur complement is equivalent to the LMI (145) of Theorem 53. This completes the proof of the theorem.

Remark 57 *Use* $\begin{bmatrix} M^{-1} & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix} = -\left\{ \begin{bmatrix} -M & 0 & 0 \\ 0 & -Q^{-1} & 0 \\ 0 & 0 & -R^{-1} \end{bmatrix} \right\}^{-1}$

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 53, it is easy to prove the following Corollary.

Corollary 58 *For the uncertain (state delayed) system (132) there exist symmetric positive-definite matrices* P, T *such that* $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ *(with* $V_k^{tot} = x_k^T P x_k + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}$ *holds for all admissible uncertainties if and only if there exist a positive scalar* $\epsilon > 0$*, a matrix* $W \in \Re^{m\times n}$ *and symmetric positive definite matrices* $S = P^{-1} \in \Re^{n\times n}$, $M = P_d^{-1} \in \Re^{m\times m}$ *such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & A_1 M & 0 & 0 \\ (AS + BW)^T & -S & 0 & (E_a S + E_b W)^T & S \\ MA_1^T & 0 & -M & ME_d^T & 0 \\ 0 & (E_a S + E_b W) & E_d M & -\epsilon I & 0 \\ 0 & S & 0 & 0 & -M \\ 0 & S & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 \end{bmatrix} < 0. \tag{154}$$

Furthermore, if matrix inequality (154) has a feasible solution in terms of the variables $\{\epsilon, W, S\}$ *then the state feedback control law* $u_k = WS^{-1}x_k = Kx_k$ *is a robustly stabilizing control law.*

10.3 GCC Analysis & Synthesis for uncertain DT systems without Delays

Open–loop DT system with uncertain dynamics but without input or state delays

$$x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k, \quad x \in \Re^n,\ u \in \Re^m \tag{155}$$

- NO INPUT DELAY $\Rightarrow B_1 = \Delta B_1 = E_h = 0$
- NO STATE DELAY $\Rightarrow A_1 = \Delta A_1 = E_d = 0$

and

$$[\Delta A\ \Delta B] = DF\,[E_a\ E_b] \tag{156}$$

the unknown (time-varying) matrix F satisfying $F^T F \le I$. The closed–loop dynamics with $u_k = Kx_k$ are

$$x_{k+1} \quad = \quad [A + BK + DF(E_a + E_b K)]\,x_k \triangleq A_C(k)x_k \tag{157}$$

with the obvious definition for the uncertain closed-loop matrix A_C .

The sufficient condition for the existence of memoryless state feedback GCC law is a special "case" of Theorem (68)

Theorem 59 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller for (155) if there exist symmetric positive definite matrix $P \in \Re^{n\times n}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$0 > \Pi \triangleq A_C^T P A_C \underbrace{-P + Q + K^T RK}_{\Lambda} = A_C^T P A_C + \Lambda < 0 \tag{158}$$

with the obvious definition for Λ and the uncertain closed–loop system matrix A_C already defined in (157). Moreover, following the same arguments as in Theorem (68), the closed-loop cost function satisfies

$$J_{cl} \le J^* \triangleq x_0^T P x_0 \le \lambda_{max}(U^T PU) \tag{159}$$

Proof 60 *Defining the "quadratic candidate Lyapunov function function*

$$V_k = x_k^T P x_k \tag{160}$$

with $P > 0$ being a SPDef matrix of appropriate dimensions $0 < P^T = P \in \Re^{n\times n}$, the forward difference $\Delta V_k = V_{k+1} - V_k$ along the trajectories of the closed-loop system (157) can be expressed as

$$\begin{aligned} \Delta V_k &= V_{k+1} - V_k = x_k^T \left(A_C^T P A_C - P\right) x_k \\ &= x_k^T \left[\Pi - (Q + K^T RK)\right] x_k \end{aligned} \tag{161}$$

From the assumption about the negative definiteness of the matrix appearing in Theorem 59, it is clear that the "wish" for $\Delta V_k < 0$ is indeed satisfied since (in that case)

$$\begin{aligned} \Delta V_k &= x_k^T \Pi x_k - x_k^T \left[Q + K^T RK\right] x_k \\ &< -x_k^T \left[Q + K^T RK\right] x_k \\ &\le -\lambda_{\min}(Q + K^T RK)\|x_k\|^2 < 0. \end{aligned} \tag{162}$$

Noting that $Q + K^T RK > 0$, the last inequality implies quadratic stability. Following the same arguments as in Theorem (68) the closed-loop cost function satisfies

$$J_{cl} = \sum_{k=0}^{\infty} x_k^T \left[Q + K^T RK\right] x_k \le V_0 = x_0^T P x_0 \triangleq J^* \tag{163}$$

The guaranteed cost J^* in (163) depends only on the initial condition, and not on the uncertainties. The deterministic approach to remove this dependence on the initial condition, is to assume that the initial state of the system (155) is arbitrary but belongs to the set $x_0 \in \Re^n : x_0 = Uu_0,\ u_0^T u_0 \le 1$ where U is a given matrix. The cost bound then obeys

$$J^* \triangleq x_0^T P x_0 \le \lambda_{max}(U^T P U) \tag{164}$$

This completes the proof of the theorem.

GCC SYNTHESIS FOR UNCERTAIN DISCRETE TIME SYSTEMS (WITHOUT DELAYS)

Theorem 61 *For the uncertain system (155) and the cost function (85) there exist symmetric positive-definite matrix $P \in \Re^{n\times n}$ such that matrix inequality (158) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrix $S = P^{-1} \in \Re^{n\times n}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & 0 & 0 & 0 \\ (AS+BW)^T & -S & (E_aS+E_bW)^T & S & W^T \\ 0 & (E_aS+E_bW) & -\epsilon I & 0 & 0 \\ 0 & S & 0 & -Q^{-1} & 0 \\ 0 & W & 0 & 0 & -R^{-1} \end{bmatrix} < 0 \tag{165}$$

Furthermore, if matrix inequality (165) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$, then the state feedback control law $u_k = WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies $J \le (d^*)\lambda_{\max}(US^{-1}U)$

Remark 62 *Note that the LMI (165) of Theorem 61, results from the "Generic" LMI (189) after removing the third row/column (involve M, A_1, E_d which are "zero" matrices), the sixth row/column (involve M, S) with M being a "zero" matrix while matrix S is already constrained via LMI (123) and the seventh row/column (involve W, N).*

Proof 63 *Starting from the matrix inequality (158) in Theorem 59 i.e.*

$$\Pi \triangleq A_C^T P A_C - P + Q + K^T R K = A_C^T P A_C + \Lambda < 0$$

can write it as "LMI" $\Lambda - A_C^T(-P^{-1})^{-1}A_C < 0$ which,

- *by Schur complement (Lemma 17),*
- *use of the defining expression $A + BK + DF(E_a + E_bK) \triangleq A_C$ for the uncertain matrix A_C and separation of the nominal from the uncertain parts,*
- *use of Lemma 5 (transforms specific LMIs into the equivalent "$G + \epsilon MM^T + \frac{1}{\epsilon}N^T RN$" form,*

is equivalent to

$$(158) \Leftrightarrow \Lambda - A_C^T(-P^{-1})^{-1}A_C < 0 \Leftrightarrow \begin{bmatrix} -P^{-1} & A_C \\ A_C^T & \Lambda \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -P^{-1} & (A+BK) \\ (A+BK)^T & \Lambda \end{bmatrix}$$
$$+\begin{bmatrix} D \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & (E_a+E_bK) \end{bmatrix} + \begin{bmatrix} 0 & (E_a+E_bK) \end{bmatrix}^T F^T \begin{bmatrix} D \\ 0 \end{bmatrix}^T < 0 \Leftrightarrow$$
$$\begin{bmatrix} -P^{-1}+\epsilon DD^T & A+BK \\ (A+BK)^T & \Lambda \end{bmatrix} + \frac{1}{\epsilon}\begin{bmatrix} 0 \\ (E_a+E_bK)^T \end{bmatrix}\begin{bmatrix} 0 & (E_a+E_bK) \end{bmatrix} < 0 \Leftrightarrow$$
$$\begin{bmatrix} -P^{-1}+\epsilon DD^T & A+BK & 0 \\ (A+BK)^T & \Lambda & (E_a+E_bK)^T \\ 0 & (E_a+E_bK) & -\epsilon I \end{bmatrix} < 0 \tag{166}$$

Introducing the variables

$$S = P^{-1}, \quad W = KP^{-1} = KS \tag{167}$$

and (performing a congruent transformation on inequality (166) by) pre- and post- multiplying both sides of it by the nonsingular, symmetric block–diagonal matrix $diag(I, P^{-1}, I) = diag(I, S, I)$, *post- multiplying:*

$$\begin{bmatrix} -S+\epsilon DD^T & A+BK & 0 \\ (A+BK)^T & \Lambda & (E_a+E_bK)^T \\ 0 & (E_a+E_bK) & -\epsilon I \end{bmatrix} diag(I,S,I) =$$

$$\begin{bmatrix} -S+\epsilon DD^T & (A+BK)S & 0 \\ (A+BK)^T & \Lambda S & (E_a+E_bK)^T \\ 0 & (E_a+E_bK)S & -\epsilon I \end{bmatrix}$$

and finally pre- multiplying:

$$diag(I,S,I)\begin{bmatrix} -S+\epsilon DD^T & (A+BK)S & 0 \\ (A+BK)^T & \Lambda S & (E_a+E_bK)^T \\ 0 & (E_a+E_bK)S & -\epsilon I \end{bmatrix} =$$

$$\begin{bmatrix} -S+\epsilon DD^T & (AS+BW) & 0 \\ (AS+BW)^T & S\Lambda S & (E_aS+E_bW)^T \\ 0 & (E_aS+E_bW) & -\epsilon I \end{bmatrix} < 0 \tag{168}$$

Using the fact that $S = S^T$, the definition of Λ from(158) and the definitions in (167), the $(2,2)$–element $S\Lambda S$ in (152) writes as

$$\begin{aligned} S\Lambda S &= S[-P+Q+K^TRK]S = S[-S^{-1}+Q+K^TRK]S \\ &= -S+SQS+W^TRW \end{aligned}$$

Now, single out the $-S$ term, and express the (remaining) term $SQS+W^TRW$ as

$$\begin{bmatrix} 0 & 0 \\ S^T & W^T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} 0 & S & 0 \\ 0 & W & 0 \end{bmatrix} = \tilde{M}^T diag(\,Q,\ R)\tilde{M}$$

with $\tilde{M} \triangleq \begin{bmatrix} 0 & S & 0 \\ 0 & W & 0 \end{bmatrix}$. LMI (168) becomes thus

$$\begin{bmatrix} -S+\epsilon DD^T & (AS+BW) & 0 \\ (AS+BW)^T & -S & (E_aS+E_bW)^T \\ 0 & (E_aS+E_bW) & -\epsilon I \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 \\ S^T & W^T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} 0 & S & 0 \\ 0 & W & 0 \end{bmatrix} < 0 \tag{169}$$

which by Schur complement is equivalent to the LMI (165) of Theorem 61. This completes the proof of the theorem.

10.4 Special case: Robust Stabilization of DT systems with norm–bounded uncertainties (no Delays - no GCC)

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 61, it is easy to prove the following Corollary.

Corollary 64 *For the uncertain system (155) there exist a symmetric positive-definite matrix P such that $\Delta V_k = V_{k+1} - V_k = x_k^T\left(A_C^TPA_C - P\right)x_k < 0$ (with $V_k = x_k^TPx_k$) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and a symmetric positive definite matrix $S = P^{-1} \in \Re^{n\times n}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & 0 \\ (AS+BW)^T & -S & (E_aS+E_bW)^T \\ 0 & (E_aS+E_bW) & -\epsilon I \end{bmatrix} < 0 \tag{170}$$

Furthermore, if matrix inequality (170) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$, then the state feedback control law $u_k = WS^{-1}x_k = Kx_k$ is a robustly stabilizing control law.

Proof 65 *Defining (as already done in (160), (161)) the "quadratic candidate Lyapunov function function*

$$V_k = x_k^T P x_k \tag{171}$$

with $P > 0$ being a SPDef matrix of appropriate dimensions $0 < P^T = P \in \Re^{n\times n}$, the "wish" for $\Delta V_k = V_{k+1} - V_k < 0$ along the trajectories of the closed-loop system (157) can be equivalently expressed as

$$\begin{gathered}
\Delta V_k = x_k^T \left(A_C^T P A_C - P\right) x_k < 0 \\
\Updownarrow \\
-P - A_C^T(-P^{-1})^{-1} A_C < 0 \\
\Updownarrow \\
\begin{bmatrix} -P^{-1} & A_C \\ A_C^T & -P \end{bmatrix} < 0 \\
\Updownarrow \\
\begin{bmatrix} -P^{-1} & A+BK \\ (A+BK)^T & -P \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & (E_a + E_b K) \end{bmatrix} \\
+ \begin{bmatrix} 0 & (E_a + E_b K) \end{bmatrix}^T F^T \begin{bmatrix} D \\ 0 \end{bmatrix}^T < 0 \\
\Updownarrow \\
\begin{bmatrix} -P^{-1} + \epsilon D D^T & A+BK \\ (A+BK)^T & -P \end{bmatrix} \\
+ \frac{1}{\epsilon} \begin{bmatrix} 0 \\ (E_a + E_b K)^T \end{bmatrix} \begin{bmatrix} 0 & (E_a + E_b K) \end{bmatrix} < 0 \\
\Updownarrow \\
\begin{bmatrix} -P^{-1} + \epsilon D D^T & A+BK & 0 \\ (A+BK)^T & -P & (E_a + E_b K)^T \\ 0 & (E_a + E_b K) & -\epsilon I \end{bmatrix} < 0
\end{gathered} \tag{172}$$

- *Introducing* $S = P^{-1}, \;\; W = KP^{-1} = KS$,
- *perform a congruent transformation on (172) via the nonsingular symmetric block–diagonal matrix* $\begin{bmatrix} I & 0 & 0 \\ 0 & P^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix} > 0,$

LMI (172) becomes

$$\begin{bmatrix} -S + \epsilon D D^T & (AS + BW) & 0 \\ (AS + BW)^T & -S & (E_a S + E_b W)^T \\ 0 & (E_a S + E_b W) & -\epsilon I \end{bmatrix} < 0. \tag{173}$$

which is LMI (170).

Hence if the above LMI has a feasible solution w.r.t the variables $\{\epsilon, W, S\}$ with $\{\epsilon > 0,\ W \in \Re^{m\times n}$ SPD matrix $S = P^{-1} \in \Re^{n\times n}$ then the state feedback control law $u_k = WS^{-1}x_k$ is a robustly stabilizing control law.

Remark 66 *If there are no uncertainties (ΔA = "0″ = ΔB) the previous result ends up to the well known state feedback synthesis via LMI i.e.* $u_k = WS^{-1}x_k$ *with* W, S *being the feasible solution to the following LMI*

$$\begin{bmatrix} -S & AS + BW \\ (AS + BW)^T & -S \end{bmatrix} < 0 \tag{174}$$

11 GCC Analysis & Synthesis for NETWORKED CONTROL SYSTEMS (NCS)

An application of section 10.1 on GCC Analysis & Synthesis for uncertain DT systems with (only) Input Delay.... Case1 on NCS with "smal" delay.....based on References [13, 14]...

Idea: NCS with varying networked induced delay less than one sampling period ("smal" delay) and no packet drops behave as DT systems with input delay.

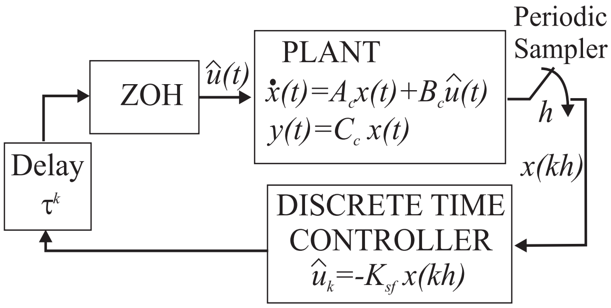


Figure 12: NCS structural framework (from [13, 14])

11.1 NCS Dynamics & Discretization

NCS Dynamics: The dynamics of the SISO–NCS under investigation is described by the combination of a continuous–time linear time–invariant plant with a discrete–time time–invariant controller [15, 16, 17, 18, 13, 14].

$$
\begin{aligned}
\dot{x}(t) &= A_c x(t) + B_c \hat{u}(t), \quad y(t) = C_c x(t) \\
& \quad t \in \left[kh + \tau^k, kh + h + \tau^{k+1}\right) \\
& \quad 0 \le \tau_{\min} < \tau^k < \tau_{\max} \le h \\
\hat{u}(t) &= \begin{cases} \hat{u}_{k-1}, & t \in \left[kh - h + \tau^{k-1},\ kh + \tau^k\right) \\ \hat{u}_k, & t \in \left[kh + \tau^k,\ kh + h + \tau^{k+1}\right). \end{cases}
\end{aligned} \tag{175}
$$

with $x \in \Re^n$ and $u, y \in \Re$ and appropriately defined initial conditions. The aggregate input delay τ^k is uncertain and time–varying with known bounds ($0 \le \tau_{\min} \le \tau^k \le \tau_{\max} \le h$). This description corresponds to a constant and known sampling period, whereas both controller and actuator are event-driven. The important modelling issue arising from (175) is that the actuation time instances are not equidistant since the piecewise constant control action $\hat{u}(t)$ experiences a "jump" at the uncertain time instance $kh + \tau^k$. Hence (unless τ^k is constant) it is not in general possible to treat the ensuing NCS in a standard sampled-data or "time–delayed" setting and a "hybrid" setup should be used ([19]). Despite the "jump" nature of $\hat{u}(t)$, the exact discretization of (175) within a sampling period is straightforward, describing the evolution of the state vector at the discrete time instances, is given by

$$
x_{k+1} = \Phi x_k + \Gamma_0(\tau^k)\hat{u}_k + \Gamma_1(\tau^k)\hat{u}_{k-1} \tag{176}
$$

where $\Phi = \exp(A_c h)$ and

$$\begin{aligned}
\Gamma_0(\tau^k) &= \int_0^{h-\tau^k} \exp(A_c\lambda) B_c d\lambda \\
&= \begin{bmatrix} I_n & \bar{0}^T \end{bmatrix} \exp\left(\begin{bmatrix} A_c & B_c \\ \bar{0} & 0 \end{bmatrix} (h-\tau^k) \right) \begin{bmatrix} \bar{0}^T \\ 1 \end{bmatrix} \\
\Gamma_1(\tau^k) &= -\Gamma_0(\tau^k) + \int_0^{h} \exp(A_c\lambda) B_c d\lambda
\end{aligned} \tag{177}$$

The decomposition $\tau^k = \tau^\circ + \tau^k_\Delta$ of the uncertain delay, with τ° denoting the user selected nominal value, leads into a corresponding system decomposition. The matrices $\Gamma_0(\tau^k)$, $\Gamma_1(\tau^k)$ can thus be decomposed into constant and known nominal parts $\bar{\Gamma}_0 \triangleq \Gamma_0(\tau_\circ)$, $\bar{\Gamma}_1 \triangleq \Gamma_1(\tau_\circ)$ and uncertain (though bounded) parts $\Delta\Gamma_0$, $\Delta\Gamma_1$ which are related as ([17])

$$\begin{aligned}
\Delta\Gamma_0(\tau^k,\tau^\circ) &= \Delta\Gamma(\tau^k,\tau^\circ) B_c = -\Delta\Gamma_1(\tau^k,\tau^\circ) \\
\text{or} \quad [\Delta\Gamma_0 \ \ \Delta\Gamma_1] &= I_n \Delta\Gamma \, [B_c \quad -B_c] \\
\text{with} \quad \Delta\Gamma(\tau^k,\tau^\circ) &\triangleq \int_{h-\tau^\circ}^{h-\tau^k} \exp(A_c\lambda) d\lambda
\end{aligned} \tag{178}$$

For $\tau^\circ = \tau_{\min}$ the "core uncertain matrix" $\Delta\Gamma(\tau^k, \tau_{\min})$ in (178) is norm bounded by ([18])

$$\begin{aligned}
\delta^s &= \sup\{\sigma_{\max}(\Delta\Gamma(\tau^k)\} \le \max_{\tau^k \in [\tau_{\min},\tau_{\max}]} \left\| \int_{h-\tau_{\min}}^{h-\tau^k} e^{A_c\lambda} d\lambda \right\|_2 \\
&\le \int_{h-\tau_{\max}}^{h-\tau_{\min}} e^{\|A_c\|_2 \lambda} d\lambda = \int_{h-\tau_{\max}}^{h-\tau_{min}} e^{\sigma_{\max}(A_c)\lambda} d\lambda \\
&= \frac{e^{\sigma_{\max}(A_c)(h-\tau_{\min})} - e^{\sigma_{\max}(A_c)(h-\tau_{\max})}}{\sigma_{\max}(A_c)} \\
&= \delta\left(\tau_{\min}, \tau_{\max}, h, A_c\right) .
\end{aligned} \tag{179}$$

Remark 67 *Any value $\delta \ge \delta^s$ can be used to bound $\sigma_{\max}(\Delta\Gamma(\tau^k))$ from above. Of particular importance is the $\delta(\tau_{\min}, \tau_{\max}, h, A_c)$-value due to its ease of computation. As a special case, setting $\tau^\circ = \tau_{\min} = 0$, the previous bound becomes a function of $\tau_{\max}$,*

$$\delta(0, \tau_{\max}, h, A_c) = \frac{e^{\sigma_{\max}(A_c)h} - e^{\sigma_{\max}(A_c)(h-\tau_{\max})}}{\sigma_{\max}(A_c)} . \tag{180}$$

It can be shown that the same expression (179) is also valid for the selection $\tau^\circ = \tau_{\max}$. Lastly recall that $\sigma_{\max}(\Delta\Gamma(\tau^k)) \le \delta \ \Leftrightarrow \Delta\Gamma^T(\tau^k)\Delta\Gamma(\tau^k) \le \delta^2 I_n$.

Combining the discretized version of NCS in (176) with the decomposition of the uncertain matrices $\Gamma_0(\tau^k)$, $\Gamma_1(\tau^k)$ presented in (178), can write

$$\begin{aligned}
x_{k+1} &= \Phi x_k + \left[\bar{\Gamma}_0 + \Delta\Gamma_0\right] u_k + \left[\bar{\Gamma}_1 + \Delta\Gamma_1\right] u_{k-1} \\
&= \Phi x_k + \left[\bar{\Gamma}_0 + \Delta\Gamma B_c\right] u_k + \left[\bar{\Gamma}_1 - \Delta\Gamma B_c\right] u_{k-1}
\end{aligned} \tag{181}$$

with $\bar{\Gamma}_0,\ \bar{\Gamma}_1$ signifying the nominal parts of $\Gamma_0(\tau^k),\ \Gamma_1(\tau^k)$.

Notice that (181) is a special case of the generic uncertain dynamics (110)

$$x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$$

in the previous chapter with

$$\begin{aligned}
h^* &\rightarrow 1 \\
A &\rightarrow \Phi \\
\Delta A &\rightarrow 0,\ \ E_a \rightarrow 0 \\
B &\rightarrow \bar{\Gamma}_0, \\
B_1 &\rightarrow \bar{\Gamma}_1, \\
\Delta B &\rightarrow \Delta\Gamma B_c, \quad \Delta B_1 \rightarrow -\Delta\Gamma B_c \\
E_b &\rightarrow B_c,\ \ E_h \rightarrow -B_c \\
D &\rightarrow I_n \\
F &\rightarrow \Delta\Gamma
\end{aligned} \tag{182}$$

Associated with the uncertain open–loop system (181) is the cost function

$$J = \sum_{k=0}^{\infty} \left[x_k^T Q x_k + u_k^T R u_k \right] \tag{183}$$

with $Q, R > 0$ being symmetric and positive definite matrices of appropriate dimensions. Assuming that the system state is available for feedback, the objective is the design of a memoryless state feedback control law $\hat{u}_k = -Kx_k$, such that for any admissible uncertainty $\Delta\Gamma(\tau^k)$ the resulting closed–loop system is not only asymptotically stable but also guarantees the satisfaction of the bound $J \leq J^*$ with the positive (constant) scalar J^* being independent of the uncertainties (see [6, 7]). The rationale behind this last objective is to incorporate (transient) performance objectives into the design procedure.For notation simplification $\hat{u}_k$ is hereafter replaced by u_k.

Closing the loop in (176), (181) with $u_k = -Kx_k$, the closed–loop dynamics can be written as

$$\begin{aligned}
x_{k+1} &= \left[\Phi - \Gamma_0(\tau^k)K\right] x_k + \left[-\Gamma_1(\tau^k)K\right] x_{k-1} \\
&= \left[\Phi - \bar{\Gamma}_0 K - \Delta\Gamma_0 K\right] x_k + \left[-\bar{\Gamma}_1 K - \Delta\Gamma_1 K\right] x_{k-1} \\
&= \left[\Phi - \bar{\Gamma}_0 K - \Delta\Gamma B_c K\right] x_k + \left[-\bar{\Gamma}_1 + \Delta\Gamma B_c\right] K x_{k-1} \\
&\triangleq A_{cl}(k) x_k + B_{cl}(k) K x_{k-1}
\end{aligned} \tag{184}$$

with the obvious definitions for the uncertain closed–loop matrices $A_{cl}(k),\ B_{cl}(k)$ i.e. $(A_{cl}(k) = \left[\Phi - \bar{\Gamma}_0 K - \Delta\Gamma B_c K\right],\ B_{cl}(k) = \left[-\bar{\Gamma}_1 + \Delta\Gamma B_c\right])$ and

$$J_{cl} = \sum_{k=0}^{\infty} x_k^T \left[Q + K^T R K\right] x_k \tag{185}$$

11.2 Guaranteed Cost Control (GCC) Analysis for NCS - a sufficient condition

The following Theorem presents a sufficient condition for the existence of memoryless state feedback guaranteed cost control law for the uncertain networked system in (181).

Theorem 68 *The control law $u_k^* = -Kx_k$ is a guaranteed cost controller if there exist symmetric positive definite matrices $P \in \Re^{n\times n}, T \in \Re$ such that for any admissible "core uncertain" matrix $\Delta\Gamma(\tau^k)$ given in (178),(179) the following inequality holds:*

$$\begin{bmatrix} \Pi & A_{cl}^T P B_{cl} \\ * & B_{cl}^T P B_{cl} - T \end{bmatrix} < 0 \tag{186}$$

with

$$\begin{aligned} \Pi &\triangleq A_{cl}^T P A_{cl} - P + K^T T K + Q + K^T R K \\ &\triangleq A_{cl}^T P A_{cl} + \Lambda, \end{aligned} \tag{187}$$

and the obvious definition for the symmetric matrix $\Lambda = \Lambda^T$, while A_{cl}, B_{cl} have been defined in (184). Moreover if (186) is true, the cost function is bound as $J^* \leq \lambda_{\max}(U^T P U) + \lambda_{\max}(U^T K^T T K U)$ with U a given matrix depending on the initial conditions.

The Proof follows the same lines as the Proof of Theorem 45 with the correspondences shown in 182 and the input delay h bounded by 1 i.e. $0 \leq h \leq h^* = 1$.

11.3 Synthesis of GCC for NCS (INCOMPLETE)

The sufficient condition in Theorem 68 will now be used for the synthesis of stabilizing Guaranteed Cost Controllers. The synthesis procedure is stated below, for NCS as an LMI feasibility problem. Lemma 5 will be used in the proof.

Theorem 69 *For the uncertain system (181) and the cost function (183) there exist symmetric positive-definite matrices P, T such that matrix inequality (186) holds for all admissible uncertainties (i.e. for any admissible "core uncertain" matrix $\Delta\Gamma(\tau^k)$ presented in (178),(179)) if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{1\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $N = T^{-1} \in \Re$ such that the following LMI is satisfied.*

$$\begin{bmatrix} (-S+\epsilon I_n) & (\Phi S - \bar{\Gamma}_0 W) & -\bar{\Gamma}_1 N & 0 & 0 & 0 & 0 \\ * & -S & 0 & -\delta(B_c W)^T & W^T & S^T & W^T \\ * & * & -N & \delta N B_c^T & 0 & 0 & 0 \\ * & * & * & -\epsilon I_n & 0 & 0 & 0 \\ * & * & * & 0 & -N & 0 & 0 \\ * & * & * & 0 & 0 & -Q^{-1} & 0 \\ * & * & * & * & * & * & -R^{-1} \end{bmatrix} < 0. \tag{188}$$

or (after a permutation)

$$\begin{bmatrix} (-S+\epsilon I_n) & 0 & -\bar{\Gamma}_1 N & 0 & 0 & (\Phi S - \bar{\Gamma}_0 W) & 0 \\ * & -Q^{-1} & 0 & 0 & W^T & S^T & W^T \\ * & * & -N & \delta N B_c^T & 0 & S & 0 \\ * & * & * & -\epsilon I_n & 0 & -\delta B_c W & 0 \\ * & * & * & * & -R^{-1} & W & 0 \\ * & * & * & * & * & -S & W^T \\ * & * & * & * & * & * & -N \end{bmatrix} < 0. \tag{189}$$

Furthermore, if matrix inequality (189) has a feasible solution in terms of the variables ϵ, W, S, N, then the state feedback control law $u_k = -WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed–loop cost function satisfies $J^* \leq 2\lambda_{\max}(U^T S^{-1} U)$.

The Proof follows the same lines as the Proof of Theorem 46 with the input delay h bounded by 1 i.e. $0 \leq h \leq h^* = 1$, up to the point where the following synthesis LMI is derived

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & B_1 N & 0 & 0 & 0 & 0 \\ * & -S & 0 & (E_a S + E_b W)^T & W^T & S & W^T \\ * & * & -N & NE_h^T & 0 & 0 & 0 \\ * & * & * & -\epsilon I & 0 & 0 & 0 \\ * & * & * & * & -N & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & * & -R^{-1} \end{bmatrix} < 0.$$

Using the correspondences shown in 182, can write...
(Note: the sign differences are due to the control law $u_k^* = -Kx_k$ used in [13, 14])

11.4 Synthesis of Robust State Feedback Control for NCS (no GCC)

This section comes from our paper [20].The main result here is a synthesis procedure of a robustly stabilizing SSF (a stabilizing gain K for all admissible $\tau^k \in [0, \tau^s_{\max}]$) and is presented as Theorem 70.

Although this result is a special case of the more generic ("GCC") LMI (188) in Theorem 69 the proof is presented "in a single step" i.e. without going through the stability analysis...

Starting again from (176),(177), the matrices $\Gamma_0(\tau^k),\ \Gamma_1(\tau^k)$ can be decomposed into constant and known nominal parts $\Gamma_0^\circ \triangleq \Gamma_0(\tau_\circ),\ \Gamma_1^\circ \triangleq \Gamma_1(\tau_\circ)$ and uncertain (though bounded) parts $\Delta\Gamma_0,\ \Delta\Gamma_1$ related as in (178) and the discrete–time open loop NCS dynamics writes as in (181) i.e.

$$x_{k+1} = \Phi x_k + \left[\Gamma_0^\circ + \Delta\Gamma_0\right] u_k + \left[\Gamma_1^\circ + \Delta\Gamma_1\right] u_{k-1} \ .$$

...the "core uncertain matrix" $\Delta\Gamma(\tau^k)$ in (178) is norm bounded and for the selection $\tau^\circ = 0$ this bound can be approximated as already presented in (180) [21]

$$\delta^s = \sup\{\sigma_{\max}(\Delta\Gamma(\tau_k)\} \leq \frac{e^{\sigma_{\max}(A_c h)} - e^{\sigma_{\max}(A_c)(h-\tau_{\max})}}{\sigma_{\max}(A_c)} = \delta\left(\tau_{\max}, h, A_c\right) \ .$$

Closing the loop in (176),(181) via a discrete–time static state feedback law ($\hat{u}_k = -Kx_k,\ \ \hat{u}_{k-1} = -Kx_{k-1}$), the closed–loop dynamics becomes (see (184))

$$\begin{aligned} x_{k+1} &= \left[\Phi - \Gamma_0(\tau^k)K\right] x_k + \left[-\Gamma_1(\tau^k)K\right] x_{k-1} \\ &= \left[\Phi - \Gamma_0^\circ K - \Delta\Gamma B_c K\right] x_k + \left[-\Gamma_1^\circ + \Delta\Gamma B_c\right] K x_{k-1} \\ &\triangleq A_{cl}(k) x_k + B_{cl}(k) K x_{k-1} \end{aligned}$$

with the obvious definitions for the uncertain closed–loop matrices $A_{cl}(k),\ B_{cl}(k)$.

Theorem 70 *For the uncertain system (181) and for all admissible uncertainties, if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{1\times n}$ and symmetric positive definite matrices $S \in \Re^{n\times n}$, $N \in \Re$ such that the following LMI has a feasible solution,*

$$\begin{bmatrix} (-S + \epsilon I_n) & (\Phi S - \Gamma_0^\circ W) & -\Gamma_1^\circ N & 0 & 0 \\ * & -S & 0 & -\delta(B_c W)^T & W^T \\ * & * & -N & \delta N B_c^T & 0 \\ * & * & * & -\epsilon I_n & 0 \\ * & * & * & * & -N \end{bmatrix} < 0. \tag{190}$$

then the state feedback control law $u_k = -WS^{-1}x_k = -Kx_k$ is a "robustly stabilizing" control law for all admissible, arbitrary time–varying uncertainties $\Delta\Gamma(\tau^k)$ presented in (178),(180), and arbitrary time–varying delays $\tau^k \in [0,\ \tau^s_{\max}]$, where $\tau^s_{\max}$ not necessarily related to $\tau_{\max}$ from the previous paragraphs.

Proof: Defining the augmented state vector $\tilde{\xi}_k \triangleq \begin{bmatrix} x_k \\ Kx_{k-1} \end{bmatrix} \in \Re^{(n+1)}$ and a candidate Lyapunov function

$$V_k = V(x_k, x_{k-1}) = x_k^T \tilde{P} x_k + x_{k-1}^T K^T T K x_{k-1}, = \tilde{\xi}_k^T \begin{bmatrix} \tilde{P} & 0 \\ 0 & T \end{bmatrix} \tilde{\xi}_k, \tag{191}$$

with $\tilde{P},\ T > 0$ being symmetric and positive definite matrices of appropriate dimensions ($\tilde{P} \in \Re^{n\times n}$), and noting that the $\tilde{\xi}_k$ dynamics obey $\tilde{\xi}_{k+1} = \begin{bmatrix} A_{cl} & B_{cl} \\ K & 0 \end{bmatrix} \tilde{\xi}_k$, the forward difference ΔV_k along the trajectories of the closed-loop system in (184), can be expressed after some calculations as

$$\Delta V_k = \tilde{\xi}_k^T \begin{bmatrix} A_{cl}^T \tilde{P} A_{cl} + \Lambda & A_{cl}^T \tilde{P} B_{cl} \\ * & B_{cl}^T \tilde{P} B_{cl} - T \end{bmatrix} \tilde{\xi}_k \tag{192}$$

where "$*$" induces symmetry as usual in the LMI literature and the symmetric matrix Λ defined as $\Lambda = -\tilde{P} + K^T T K = \Lambda^T$. Using Schur complements the demand for $\Delta V_k < 0$ is equivalent to

$$\begin{bmatrix} \Lambda & 0 \\ 0 & -T \end{bmatrix} + \begin{bmatrix} A_{cl}^T \\ B_{cl}^T \end{bmatrix} \tilde{P} \begin{bmatrix} A_{cl} & B_{cl} \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -\tilde{P}^{-1} & A_{cl} & B_{cl} \\ A_{cl}^T & \Lambda & 0 \\ B_H^T & 0 & -T \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} -\tilde{P}^{-1} & \Phi - \Gamma_0^\circ K & -\Gamma_1^\circ \\ (\Phi - \Gamma_0^\circ K)^T & \Lambda & 0 \\ (-\Gamma_1^\circ)^T & 0 & -T \end{bmatrix} + \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} \Delta\Gamma(\tau^k) \begin{bmatrix} 0 & (-B_c K) & B_c \end{bmatrix} +$$
$$\begin{bmatrix} 0 \\ (-B_c K)^T \\ (B_c)^T \end{bmatrix} \Delta\Gamma^T(\tau^k) \begin{bmatrix} I_n & 0 & 0 \end{bmatrix} < 0 \tag{193}$$

where the defining expressions of the uncertain matrices A_{cl}, B_{cl} from (184) have been used. Using the computable norm bound δ of the "core uncertain matrix" $\Delta\Gamma(\tau^k)$ (see (180) and $\Delta\Gamma^T(\tau^k)\Delta\Gamma(\tau^k) \le \delta^2 I_n$), Lemma 5 is now invoked to transform (193) it into the following equivalent matrix inequality where $\epsilon > 0$

$$\begin{bmatrix} -\tilde{P}^{-1} + \epsilon I_n & \Phi - \Gamma_0^\circ K & -\Gamma_1^\circ \\ (\Phi - \Gamma_0^\circ K)^T & \Lambda & 0 \\ (-\Gamma_1^\circ)^T & 0 & -T \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} 0 \\ (-\delta B_c K)^T \\ (\delta B_c)^T \end{bmatrix} \begin{bmatrix} 0 & -\delta B_c K & \delta B_c \end{bmatrix} < 0$$

which by Schur Complement is equivalent to

$$\begin{bmatrix} -\tilde{P}^{-1} + \epsilon I_n & \Phi - \Gamma_0^\circ K & -\Gamma_1^\circ & 0 \\ * & \Lambda & 0 & (-\delta B_c K)^T \\ * & * & -T & (\delta B_c)^T \\ * & * & * & -\epsilon I_n \end{bmatrix} < 0 \tag{194}$$

Pre- and post- multiplying both sides of (194)by the nonsingular, symmetric block–diagonal matrix $\text{diag}(I_n, \tilde{P}^{-1},\ T^{-1}, I_n)$, a congruent transformation, while introducing $S = \tilde{P}^{-1}$, $N = T^{-1}$, $W =$

$K\tilde{P}^{-1} = KS$, (194) is transformed into:

$$\begin{bmatrix} -S+\epsilon I & \Phi S - \Gamma_0^\circ W & -\Gamma_1^\circ N & 0 \\ * & S\Lambda S & 0 & -\delta(B_c W)^T \\ * & * & -N & \delta N(B_c)^T \\ * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{195}$$

Recalling the definition $\Lambda = -\tilde{P}^{-1} + K^T T K = \Lambda^T$ and the fact that $S = S^T$, the $(2,2)$–element of the last inequality is equal to

$$S\Lambda S \quad = \quad \tilde{P}^{-1}[-\tilde{P} + K^T T K]\tilde{P}^{-1} = -S + W^T N^{-1} W$$

and hence (195) writes as

$$\begin{bmatrix} -S+\epsilon I & \Phi S - \Gamma_0^\circ W & -\Gamma_1^\circ N & 0 \\ * & -S & 0 & -\delta(B_c W)^T \\ * & * & -N & \delta N(B_c)^T \\ * & * & * & -\epsilon I \end{bmatrix} + \begin{bmatrix} 0 \\ W^T \\ 0 \\ 0 \end{bmatrix} N^{-1} \begin{bmatrix} 0 & W & 0 & 0 \end{bmatrix} < 0$$

which by Schur complement is equivalent to (190) which is an LMI in terms of the variables ϵ, W, S, N. This completes the proof of the theorem. □

12 An Alternative GCC Analysis & Synthesis for uncertain DT systems with (only) State Delay (Guan et.al. IEE 1999)

Presentation is primarily based on [6] i.e. the paper by X. Guan, Z. Lin and G. Duan "Robust guaranteed cost control for discrete-time, uncertain systems with delay", IEE Proc.-Control Theory Appl., vol. 146, November 1999, p.598–602.

12.1 Open–Loop GCC Analysis for systems with (only) State Delay

Open–loop DT system with (only) state delay and uncertain dynamics

$$\begin{aligned} x_{k+1} &= (A+\Delta A)x_k + (A_1+\Delta A_1)x_{k-d} + (B+\Delta B)u_k \\ &\stackrel{\Delta}{=} \tilde{A}x_k + \tilde{A}_1 x_{k-d} + \tilde{B}u_k \\ x_k &= \phi(k), \quad -d \le k \le 0 \end{aligned} \tag{196}$$

with $x \in \Re^n$ and $u \in \Re^m$ and d being a positive integer (?? unknown constant ??) integer (delay units in the state).

The system uncertainties are norm–bounded

$$\Delta A = H_1 F_1 E_1, \quad \Delta A_1 = H_2 F_2 E_2 \tag{197}$$

with the unknown (time-varying) matrices F_i satisfying $F_i^T F_i \le I, \ i = 1, 2$.

We shall need the matrix inequality presented in Lemma (11) (inequality (10))

Lemma 71 *Let A, M, N, Δ be real matrices of appropriate dimensions with $||\Delta||_2 < 1$. Then for $P > 0$ and scalar $\varepsilon > 0$ satisfying $\varepsilon I - M^T PM > 0$,*

$$(A + M\Delta N)^T P(A + M\Delta N) \le A^T[P^{-1} - \varepsilon^{-1}MM^T]^{-1}A + \varepsilon N^T N \tag{198}$$

and/or the alternative formulation of Lemma (12).

Lemma 72 *Let A, M, N, Δ be real matrices of appropriate dimensions with $||\Delta||_2 < 1$. Then for $P > 0$ and scalar $\epsilon > 0$ satisfying $P - \epsilon MM^T > 0$*

$$(A + M\Delta M)^T P^{-1}(A + M\Delta N) \le A^T[P - \epsilon MM^T]^{-1}A + \frac{1}{\epsilon}N^T N \tag{199}$$

and also the following Lemma

Lemma 73 *For any $z, y \in \Re^n$ and for any positive definite $\in \Re^{n\times n}$*

$$2z^T Py \le z^T Pz + y^T Py \tag{200}$$

Associated with the uncertain unforced open–loop system (196) is the cost function

$$J = \sum_{k=0}^{\infty} \left[x_k^T Q x_k\right] \tag{201}$$

12.2 Sufficient condition for Robust GCC Stability of the Open Loop system

THE FOLLOWING THEOREM (FORMULATED WITH < SIGN) STATES THE SUFFICIENT CONDITION FOR...

Theorem 74 *A matrix $P > 0$ is a "quadratic cost matrix" for the unforced (open–loop) system (196) and the cost function (201) if there exist parameters $\epsilon_i > 0, \quad i = 1, 2$ such that the following LMI is satisfied*

$$\begin{bmatrix} -\frac{1}{2}P + \frac{1}{2}Q + \frac{1}{\epsilon}E_1^T E_1 + \frac{1}{\epsilon}E_2^T E_2 & A^T & A_1^T \\ A & -P^{-1} + \epsilon_1 H_1 H_1^T & 0 \\ A_1 & 0 & -P^{-1} + \epsilon_2 H_2 H_2^T \end{bmatrix} < 0 \tag{202}$$

Proof 75 *Defining*

- *the "positive with respect to x_k" function (it is same one used in (139) Theorem 51)*

$$V_k^{tot} = V_k^1 + V_k^2 = x_k^T P x_k + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i} \tag{203}$$

  with $P,\ P_d > 0$ being SPDef matrices of appropriate dimensions $0 < P,\ P_d \in \Re^{n\times n}$,

- *the augmented state vector $\xi_k \triangleq \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix} \in \Re^{2n}$ which allows to write the open-loop dynamics (196) as*
  $x_{k+1} = \tilde{A}x_k + \tilde{A}_1 x_{k-d} = \begin{bmatrix} \tilde{A} & \tilde{A}_1 \end{bmatrix} \xi_k$

the forward difference $\Delta V_k = V_{k+1}^{tot} - V_k^{tot}$ along the trajectories of (196) can be expressed in terms of ξ_k as follows:

ΔV_k^1–term:

$V_k^1 = x_k^T P x_k$ *can be written as* $\xi_k^T \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \xi_k$

Using $x_{k+1} = [A_C \ \ A_D]\,\xi_k$*, the* V_{k+1}^1 *term writes as*

$$\begin{aligned} V_{k+1}^1 &= x_{k+1}^T P x_{k+1} = \xi_k^T \begin{bmatrix} \tilde{A} & \tilde{A}_1 \end{bmatrix}^T P \begin{bmatrix} \tilde{A} & \tilde{A}_1 \end{bmatrix} \xi_k \\ &= \xi_k^T \begin{bmatrix} \tilde{A}^T P \tilde{A} & \tilde{A}^T P \tilde{A}_1 \\ * & \tilde{A}_1{}^T P \tilde{A}_1 \end{bmatrix} \xi_k \end{aligned}$$

and

$$\Delta V_k^1 = V_{k+1}^1 - V_k^1 = \xi_k^T \begin{bmatrix} \tilde{A}^T P \tilde{A} - P & \tilde{A}^T P \tilde{A}_1 \\ * & \tilde{A}_1{}^T P \tilde{A}_1 \end{bmatrix} \xi_k \tag{204}$$

ΔV_k^2–term:

$$\begin{aligned} \Delta V_k^2 &= \left[\sum_{i=1}^{d} x_{k+1-i}^T P_d x_{k+1-i} \right] - \left[\sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i} \right] \\ &= x_k^T P_d x_k - x_{k-d}^T P_d x_{k-d} \\ &= \xi_k^T \begin{pmatrix} P_d & 0 \\ 0 & -P_d \end{pmatrix} \xi_k \end{aligned} \tag{205}$$

Combining (205),(204) and defining

$$\Pi_1 \triangleq \tilde{A}^T P\tilde{A} - P + P_d + Q \triangleq \tilde{A}^T P\tilde{A} + \Lambda \tag{206}$$

(?? see also the definition (137) for the forced version ??)
we have

$$\begin{aligned}\Delta V_k^{tot} &= \Delta V_k^1 + \Delta V_k^2\\ &= \xi_k^T \begin{bmatrix} \tilde{A}^T P\tilde{A} - P + P_d & \tilde{A}^T P\tilde{A}_1 \\ * & \tilde{A}_1{}^T P\tilde{A}_1 - P_d \end{bmatrix} \xi_k\\ &\triangleq \xi_k^T \begin{bmatrix} \Pi_1 - Q & \Pi_2 \\ * & \Pi_3 \end{bmatrix} \xi_k\end{aligned}$$

Following the same arguments as in sections 10.2,10.2.1, we have the ***sufficient condition*** *that the negative definiteness of the uncertain open–loop matrix*

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} < 0 \tag{207}$$

implies that the "wish" for $\Delta V_k^{tot} < 0$ *is indeed satisfied since (in that case)*

$$\begin{aligned}\Delta V_k^{tot} &= \xi_k^T \left(\begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} - \begin{bmatrix} Q & 0 \\ * & 0 \end{bmatrix} \right) \xi_k\\ &\leq -x_k^T Q x_k \leq -\lambda_{\min}(Q)\|x_k\|^2 < 0.\end{aligned}$$

and

$$J_{ol} = \sum_{k=0}^{\infty} x_k^T Q x_k \leq V_0^{tot} = x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} \triangleq J^* \tag{208}$$

i.e. Stability with "guaranteed cost".

Now the ***sufficient condition*** *(207)* $W_1 \triangleq \xi_k^T \begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} \xi_k < 0$ *using Lemmas (71), (72), (73) and the definitions of* Π_1, Π_2, Π_3 *from (206)*

$$\begin{aligned}\Pi_1 &\triangleq \tilde{A}^T P\tilde{A} - P + P_d + Q \triangleq \tilde{A}^T P\tilde{A} + \Lambda\\ \Pi_2 &\triangleq \tilde{A}^T P\tilde{A}_1\\ \Pi_3 &\triangleq \tilde{A}_1{}^T P\tilde{A}_1 - P_d\end{aligned}$$

writes as

$$W_1 \triangleq \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix}^T \begin{bmatrix} \Pi_1 & \Pi_2 \\ * & \Pi_3 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-d} \end{bmatrix} = x_k^T \Pi_1 x_k + 2x_k^T \Pi_2 x_{k-d} + x_{k-d}^T \Pi_3 x_{k-d} < 0$$

Bounding procedure for the three terms of W_1.

[1] The Π_1 ***term:*** *Using Lemma (71) on the* Π_1 *term*

$$\begin{aligned}\Pi_1 &\triangleq \tilde{A}^T P\tilde{A} - P + P_d + Q \triangleq (A + H_1F_1E_1)^T P(A + H_1F_1E_1) + (P_d - P + Q)\\ &\leq \left[A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T \right)^{-1} A + \frac{1}{\epsilon_1} E_1^T E_1 \right] + (P_d - P + Q)\end{aligned}$$

hence

$$x_k^T \Pi_1 x_k \leq x_k^T \left(\left[A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T\right)^{-1} A + \frac{1}{\epsilon_1} E_1^T E_1 \right] + (P_d - P + Q) \right) x_k$$

***[2] The** Π_2 **term:** Using Lemma (73) i.e.* $2z^T P y \leq z^T P z + y^T P y$ *on the expression* $2x_k^T (A + H_1 F_1 E_1)^T P (A_1 + H_2 F_2 E_2) x_{k-d}$ *,*

and subsequently Lemma (71) on the derived expressions $(A + H_1 F_1 E_1) P (A + H_1 F_1 E_1)$ *and* $(A_1 + H_2 F_2 E_2) P (A_1 + H_2 F_2 E_2)$ *below, can bound the* $2x_k^T \Pi_2 x_{k-d}$ *term as*

$$2x_k^T \Pi_2 x_{k-d} = 2x_k^T (\tilde{A}^T P \tilde{A}_1) x_{k-d} = 2x_k^T (A + H_1 F_1 E_1)^T P (A_1 + H_2 F_2 E_2) x_{k-d}$$

$$\leq x_k^T (A + H_1 F_1 E_1)^T P (A + H_1 F_1 E_1) x_k + x_{k-d}^T (A_1 + H_2 F_2 E_2)^T P (A_1 + H_2 F_2 E_2) x_{k-d}$$
$$\leq x_k^T \left[A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T\right)^{-1} A + \frac{1}{\epsilon_1} E_1^T E_1 \right] x_k + x_{k-d}^T \left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] x_{k-d}$$

***[3] The** Π_3 **term:** Using Lemma (71) on* Π_3

$$\begin{aligned} \Pi_3 \triangleq \tilde{A}_1{}^T P \tilde{A}_1 - P_d &\triangleq (A_1 + H_2 F_2 E_2)^T P (A_1 + H_2 F_2 E_2) - P_d \\ &\leq \left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] - P_d \end{aligned}$$

hence

$$x_{k-d}^T \Pi_3 x_{k-d} \leq x_{k-d}^T \left(\left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] - P_d \right) x_{k-d}$$

Combining results can bound W_1 *as*

$$\begin{aligned} W_1 \leq x_k^T &\left(2 \left[A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T\right)^{-1} A + \frac{1}{\epsilon_1} E_1^T E_1 \right] + (P_d - P + Q) \right) x_k \\ &+ x_{k-d}^T \left(2 \left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] - P_d \right) x_{k-d} \end{aligned} \tag{209}$$

***HERE COMES THE LIGHT !!!** Instead of keep treating* P_d *as a matrix variable for a future LMI, select*

$$P_d = 2 \left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] \tag{210}$$

so as to eliminate the last term in (209). Hence the "wish" for $W_1 < 0$ *using the special selection of* P_d *in (210) writes as*

$$W_1 \leq x_k^T \left(2 \left[A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T\right)^{-1} A + \frac{1}{\epsilon_1} E_1^T E_1 \right] + 2 \left[A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_2^T E_2 \right] + (Q - P) \right) x_k$$

and the wish for negative definiteness of the matrix

$$A^T \left(P^{-1} - \epsilon_1 H_1 H_1^T\right)^{-1} A + A_1^T \left(P^{-1} - \epsilon_2 H_2 H_2^T\right)^{-1} A_1 + \frac{1}{\epsilon_1} E_1^T E_1 + \frac{1}{\epsilon_1} E_2^T E_2 + \frac{1}{2}(Q - P) < 0$$

is by Schur Complement equivalent to

$$\begin{bmatrix} A_1^T\left(P^{-1}-\epsilon_2 H_2 H_2^T\right)^{-1}A_1+\frac{1}{\epsilon_1}E_1^TE_1+\frac{1}{\epsilon_1}E_2^TE_2+\frac{1}{2}(Q-P) & A^T \\ A & -\left(P^{-1}-\epsilon_1 H_1 H_1^T\right) \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} \frac{1}{\epsilon_1}E_1^TE_1+\frac{1}{\epsilon_1}E_2^TE_2+\frac{1}{2}(Q-P) & A^T \\ A & -\left(P^{-1}-\epsilon_1 H_1 H_1^T\right) \end{bmatrix} +$$

$$\begin{bmatrix} A_1^T \\ 0 \end{bmatrix}\left(P^{-1}-\epsilon_2 H_2 H_2^T\right)^{-1}\begin{bmatrix} A_1 & 0 \end{bmatrix} < 0 \Leftrightarrow$$

$$\begin{bmatrix} -\frac{1}{2}P+\frac{1}{2}Q+\frac{1}{\epsilon}E_1^TE_1+\frac{1}{\epsilon}E_2^TE_2 & A^T & A_1^T \\ A & -P^{-1}+\epsilon_1 H_1 H_1^T & 0 \\ A_1 & 0 & -P^{-1}+\epsilon_2 H_2 H_2^T \end{bmatrix} < 0$$

which is the LMI appearing in Theorem (74).

13 D-Stability & LMI Regions (INCOMPLETE)

13.1 D-Stability and Pole Placement in LMI regions

Material for this section comes mainly from [22, 23, 24, 25, 1, 26] The seminal papers are: "Pole Assignment for Uncertain Systems in a Specified Disk by Output Feedback" by G. Garcia and J. Bernussou [23, 24, 27], and the papers by M. Chilali, P. Gahinet and P. Apkarian [9, 25] ("Robust Pole Placement in LMI Regions"). See also the Lecture notes by Carsten Scherer [1] "LMI's in Controller Analysis and Synthesis", available at
http://www.dcsc.tudelft.nl/~cscherer/lmi.html

Notation:

- $\otimes$ = Kronecker product
- $\mathbb{C}$ = the complex domain

Definition 76 *[1] For a real symmetric $2m \times 2m$ matrix P the set of complex numbers*

$$L_P \triangleq \left(z \in \mathbb{C} : \begin{pmatrix} I \\ zI \end{pmatrix}^* P \begin{pmatrix} I \\ zI \end{pmatrix} < 0 \right) \tag{205}$$

is called an LMI region ($*$ signifies conjugate transpose).

Definition 77 *[9, 25] An LMI region is any subset D of the complex plane that can be defined as*

$$D = \{ z \in \mathbb{C} : L + zM + \bar{z}M^T < 0 \} \tag{206}$$

where L, M are real matrices such that $L = L^T$. The matrix–valued function $f_D = L + zM + \bar{z}M^T$ is called the characteristic function of D.

Remark 78 *?? LDRI QUESTION ?? : (205, 206) are equivalent provided that $P = \begin{pmatrix} L & M \\ M^T & 0 \end{pmatrix}$*

Example 79 ***Disk centered at (-q,0) with radius r as LMI Region***

$$|z+q|^2 < r^2 \Leftrightarrow (z+q)(\bar{z}+q) - r^2 < 0 \Leftrightarrow -r + (z+q)(r^{-1})(\bar{z}+q) < 0$$

$$\Leftrightarrow \begin{pmatrix} -r & z+q \\ \bar{z}+q & -r \end{pmatrix} < 0 \Leftrightarrow \begin{pmatrix} -r & q \\ q & -r \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \bar{z} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} < 0$$

or $P_{disc} = \begin{pmatrix} -r^2 & 0 \\ 0 & 1 \end{pmatrix}$ *according to Definition (205).*

***Subcase**: Disk centered at $(0,0)$ with radius r, writes as $z\bar{z} < r^2$.*

Example 80 ***Half-plane:*** *$Re(z) < \alpha \Leftrightarrow z + \bar{z} - 2\alpha < 0$ or $P_{halfplane} = \begin{pmatrix} -2\alpha & 1 \\ 1 & 0 \end{pmatrix}$ according to Definition (205).*

Example 81 ***Sector:*** *$Re(z)tan(\phi) < -|Im(z)|$ or $P_{sector} = \begin{pmatrix} 0 & 0 & sin(\phi) & cos(\phi) \\ 0 & 0 & -cos(\phi) & sin(\phi) \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}$ according to Definition (205).*

Remark 82 *[1] The intersection of finitely many LMI regions is an LMI region. (Can hence describe quite general subsets of $\mathbb{C}$. Used later for closed-loop pole-placement with LMI techniques.)*
Proof *follows from usual stacking property of LMI's:*

$$\begin{pmatrix} I \\ zI \end{pmatrix}^* \begin{pmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{pmatrix} \begin{pmatrix} I \\ zI \end{pmatrix} < 0, \begin{pmatrix} I \\ zI \end{pmatrix}^* \begin{pmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{pmatrix} \begin{pmatrix} I \\ zI \end{pmatrix} < 0$$
$$\textit{if and only if} \begin{pmatrix} I & 0 \\ 0 & I \\ zI & 0 \\ 0 & zI \end{pmatrix}^* \begin{pmatrix} Q_1 & 0 & S_1 & 0 \\ 0 & Q_2 & 0 & S_2 \\ S_1^T & 0 & R_1 & 0 \\ 0 & S_2^T & 0 & R_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ zI & 0 \\ 0 & zI \end{pmatrix} \tag{207}$$

General Stability Characterization in terms of LMI regions [1]

Theorem 83 *[1] All eigenvalues of $A \in R^{n\times n}$ are contained in the LMI region*

$$\begin{pmatrix} I \\ zI \end{pmatrix}^* \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} I \\ zI \end{pmatrix} < 0$$

if and only if *there exists a $K > 0$ such that*

$$\begin{pmatrix} I \\ A \otimes I \end{pmatrix}^* \begin{pmatrix} K \otimes Q & K \otimes S \\ K \otimes S^T & K \otimes R \end{pmatrix} \begin{pmatrix} I \\ A \otimes I \end{pmatrix} < 0 \tag{208}$$

Corollary 84 *[1]*
Discrete-Time versus Continuous-Time Stability Criteria in terms of LMI regions.

{ *The unit disk* } { *The open left half-plane* } *are LMI regions:*

$$\begin{pmatrix} 1 \\ z \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ zI \end{pmatrix} < 0, \quad \begin{pmatrix} 1 \\ z \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ zI \end{pmatrix} < 0 \quad \textit{respectively}$$

{ *A in $x_{k+1} = Ax_k$ is Schur matrix* } { *A in $\dot{x}(t) = Ax(t)$ is Hurwitz matrix* } ***if and only if*** *there exists a $K > 0$ such that*

$$\begin{pmatrix} I \\ A \end{pmatrix}^* \begin{pmatrix} -K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} < 0, \quad \begin{pmatrix} I \\ A \end{pmatrix}^* \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} < 0 \quad \textit{respectively}$$

13.2 A special case: Disk Stability & SSF Synthesis for Disc Stability

Objective: Given the "original" CT system

$$\begin{aligned} \Sigma_{OCT}: \quad \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{209}$$

design a SSF $u(t) = Kx(t)$ so that the closed–loop poles lie within a prespecified disc $\mathbb{C}(d, r)$ i.e. a disk centered at $\alpha = -(d + r) < 0$ with radius r as shown in the Figure 13 below.

Remark 85 ***NOTE CAREFULLY THE MEANING OF THE SYMBOLS*** $d, r, \alpha = -(d+r) < 0$ ***USED IN THIS PRESENTATION...MANY MISUNDERSTANDINGS MAY OCCUR DUE TO THE FACT THAT IN SOME PAPERS THE DISK CENTER*** a ***IS POSITIVE AND NEGATIVE IN OTHERS...e.g ELSEWHERE CAN MEET THE NOTATION*** $\mathbb{C}(-\alpha, r)$ ***WHERE*** α ***HAS DIFFERENT MEANING***

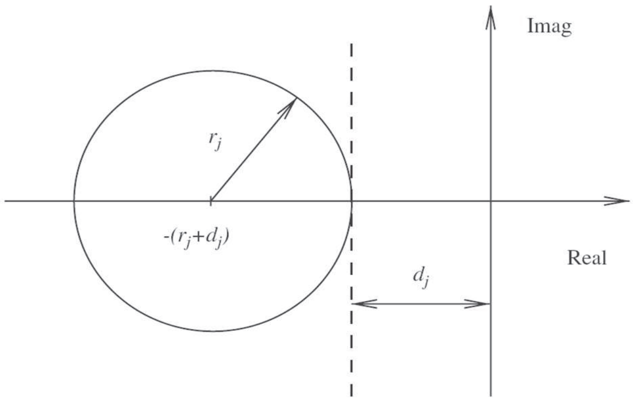


Figure 13: Disc-stability notion: the notation $\mathbb{C}(d, r)$ signifies a disk centered at $\alpha = -(d+r) < 0$ with radius r(adapted from [26])

The next two Lemmas will be used in the proof of the fundamental Theorem 88

Lemma 86 *(Given* $\alpha, r \in \mathbb{R}$*), the mapping* $z = \frac{s-\alpha}{r} \Leftrightarrow s = rz + \alpha = r(z + \frac{\alpha}{r})$ *maps a disk* $\mathbb{C}(d, r)$ *in the "s–Domain" (Laplace) into the unit circle* $\mathbb{C}(0, 1)$ *in the "z–Domain"*

Proof of Lemma 86: A disk in the complex "s-domain" centered at $\alpha = -(d+r) < 0$ with radius r (see Figure 13) is expressed as $|s - \alpha| < |r| = r \Leftrightarrow |\frac{s-\alpha}{r}| < 1 \Leftrightarrow |z| < 1$ by definition of z.

Lemma 87 *If* λ_r *is an eigenvalue of matrix* $A_r \triangleq \frac{1}{r}(A - \alpha I)$ *then* $\lambda = r\lambda_r + \alpha$ *is an eigenvalue of matrix* A*.*

Proof of Lemma 87: Assume λ_r is an eigenvalue of matrix $A_r \triangleq \frac{1}{r}(A - \alpha I)$ and x_r the corresponding eigenvector. Then $A_r x_r = \lambda_r x_r$ or $\frac{1}{r}(A - \alpha I)x_r = \lambda_r x_r \Leftrightarrow Ax_r = (r\lambda_r + \alpha)x_r$. Setting $\lambda \triangleq r\lambda_r + \alpha$ it is obvious that $\lambda = r\lambda_r + \alpha$ is an eigenvalue of matrix A.

A fundamental result (**necessary and sufficient condition**) is the following [22, 23, 24, 27]

Theorem 88 *The Disk stability/stabilization of the CT system* Σ_{OCT} $\dot{x}(t) = Ax(t) + Bu(t)$ *within* $\mathbb{C}(d, r)$*, is* ***equivalent*** *with the stability/stabilization of the following "Virtual Discrete Time System":*

$$\begin{aligned} \Sigma_{VDT}: \quad x_r[k+1] &= A_r x_r[k] + B_r u_r[k] \\ y_r[k] &= C_r x_r[k] + D_r u_r[k] \end{aligned} \tag{210}$$

with

$$A_r \triangleq \frac{1}{r}(A - \alpha I), \quad B_r \triangleq \frac{1}{\sqrt{r}}B, \quad C_r \triangleq \frac{1}{\sqrt{r}}C, \quad D_r \triangleq D \tag{211}$$

Moreover if K_r is a stabilizing feedback gain for the "Virtual Discrete Time System" Σ_{VDT} in (210) placing the eigenvalues of the virtual closed–loop matrix $A_r^{cl} = A_r + B_r K_r$ inside the unit disk $\mathbb{C}(0,1)$, then the feedback gain

$$K = \sqrt{r} K_r \tag{212}$$

places the eigenvalues of the original closed–loop matrix $A^{cl} = A + BK$ inside the disk $\mathbb{C}(d,r)$.

Sketch of Proof for Theorem 88: The proof **regarding stability** is a direct consequence of the two previous Lemmas. Due to the specific structure of $A_r \triangleq \frac{1}{r}(A - \alpha I)$ in (211) then, according to Lemma 87, the eigenvalues λ of A satisfy $\lambda = r\lambda_r + \alpha$. If all eigenvalues λ_r of A_r lie within the unit circle, i.e. $|\lambda_r| < 1$, that would be **equivalent** with the following constraint for the eigenvalues λ of A: $\lambda = r\lambda_r + \alpha \Rightarrow |\lambda - \alpha| = r|\lambda_r| < r$, since $r > 0$ and $|\lambda_r| < 1$. The inequality $|\lambda - \alpha| < r$ means that λ lies within a disk of radius r centered at α.

Regarding stabilization: Assuming now that K_r is a stabilizing feedback gain for the "Virtual Discrete Time System" Σ_{VDT} in (210) means that the control law $u_r[k] = K_r x_r$ places the eigs of the closed–loop matrix $A_r^{cl} = A_r + B_r K_r$ inside the unit circle. Using (211),(212) can then write:

$$\begin{aligned} A_r^{cl} &= A_r + B_r K_r = \frac{1}{r}(A - \alpha I) + (\frac{1}{\sqrt{r}}B)(\frac{1}{\sqrt{r}}K) \\ &= \left(\frac{1}{r}\right)[A + BK - \alpha I] = \left(\frac{1}{r}\right)\left[A^{cl} - \alpha I\right] \end{aligned}$$

which is a relation between A_r^{cl}, A^{cl} completely analogous to the relation $A_r \triangleq \frac{1}{r}(A - \alpha I)$ assumed for the open–loop matrices. Hence their eigenvalues satisfy again the relation $\lambda^{cl} = r\lambda_r^{cl} + \alpha$.

Conclusion: If then K_r is designed so that all eigenvalues λ_r^{cl} of $A_r^{cl} = A_r + B_r K_r$ lie within the unit circle ("stabilization of the Virtual Discrete Time System Σ_{VDT}") this is equivalent with placing the eigenvalues λ^{cl} of $A^{cl} = A + BK$ inside a disk of radius r centered at α.

Remark 89 *With s, z satisfying $z = \frac{s-\alpha}{r}$ (Lemma 86) and $\{A, B, C, D\}$, $\{A_r, B_r, C_r, D_r\}$ satisfying (211), can show by direct substitution that the transfer functions of the two systems $\Sigma_{OCT} : \{A, B, C, D\}$ and $\Sigma_{VDT} : \{A_r, B_r, C_r, D_r\}$ are "equal" (in form). Indeed,*

$$\begin{aligned} G_{OCT}(s) &= C(sI - A)^{-1}B + D = C\left[r(z + \frac{\alpha}{r})I - r\frac{1}{r}A\right]^{-1} B + D \\ &= C\left[r\left[(z + \frac{\alpha}{r})I - \frac{1}{r}A\right]\right]^{-1} B + D \\ &= \frac{1}{r}C\left[(z + \frac{\alpha}{r})I - \frac{1}{r}A\right]^{-1} B + D \\ &= (\frac{1}{\sqrt{r}})C\left[zI - \frac{1}{r}(A - \alpha I)\right]^{-1}(\frac{1}{\sqrt{r}})B + D \\ &\triangleq C_r\left[zI - A_r\right]^{-1} B_r + D_r \triangleq G_{VDT}(z) \end{aligned}$$

13.3 Algorithm & MATLAB code11 for Disc Stabilization via SSF

Algorithm

1. Given the CT open loop system $\{A, B, C, D\}$ and the desired Disk $\mathbb{C}(d, r)$ with $d,\ r$ positive real numbers
2. compute $\alpha = -(d+r) < 0$ and the virtual system's matrices $\{A_r, B_r, C_r, D_r\}$ from (211) matrices
3. design a stabilizing control law $u_r[k] = K_r x_r$ for the "Virtual Discrete Time System" Σ_{VDT} in (210) (can use either "lqr" or "place")
4. then the control law $u = Kx$ with gain $K = \sqrt{r}K_r$ places the eigenvalues of the CT closed loop system within the disk $\mathbb{C}(d, r)$

MATLAB code

```
clc; clear all;  close all %
disp('DRITSAS   18-June-08   Revisited 17Jan2011   ')

disp('===================================================================')
disp('    pick a DELAY-FREE SYSTEM in  CTRB_CANONICAL form    ')
disp('===================================================================')
[A,B,C,D,sysnum,xinit,h]= pick2ndOrderTs_CTRB_CANONICAL % pick2ndOrderTs
x0=[1 ; 100 ]   % xinit

%------------- SS  OF THE CT SYSTEM
open_orig_ctsys = ss(A, B, C, D);

%------------- VERIFICATION1  ----------------------------
[Plantnum,Plantden]=ss2tf(A, B, C, D);
disp('VERIFICATION1:original CT_PLANT_TF= ');
Ptf=tf(Plantnum,Plantden) disp('open loop   eig/poles')
disp('open--loop   eigs'); eig(A) %
disp('open--loop poles');pole(Ptf)

%%--------- DIMENSIONS --------------------------------
%   nx = STATE-DIM, nu =INPUT-DIM OF THE  ORIGINAL CT SYSTEM
%%-----------------------------------------------------
nx = max(size(A));  nu = size(B) * [0 ; 1]; [nx nx]=size(A);

%------------------------------------------------------------
% CONTROLLABILITY CHECK of (A,B)
%------------------------------------------------------------
fprintf('\n\n'); disp('CONTROLLABILITY CHECK of (Ac,Bc)')

if (rank(ctrb(A,B)) ~= nx)
    CONTROLLABILITY_CT =0
    error('LDRI1: System (Ac,Bc)is NOT Controllable...Good Bye!!!!');
else
    CONTROLLABILITY_CT =1
```

```
    disp('LDRI1: System (Ac,Bc) is Controllable...Proceed');
end

%-----------------  D-STABILITY  Specifications -----------------%%
disp('    D-STABILITY   ') %
disp('alpha_ct = DESIRED Center  of  DISC for the closed CT sys
--> must be  a negative real number ')%
disp('radius_ct = DESIRED radius of   DISC -->  must be  a
positive real number ') %
alpha_ct = -5.0 radius_ct = 1.0

disp('=================================================================')
disp('    VIRTUAL  DT  SYS = {Ar,Br,Cr,Dr} to be stabilized via
DLQR  or  PLACE     ') disp('Ar = ( A - alpha*eye(nx) )/r is the
VIRTUAL  DT OPEN-LOOP matrix')
disp('=================================================================')
Ar = (A - alpha_ct*eye(nx) )/radius_ct; %
Br = B/sqrt(radius_ct) ;%
Cr = C/sqrt(radius_ct) ; %
Dr = D/radius_ct ;

disp('=================================================================')
disp('     DLQR COMPUTATIONS  for the VIRTUAL DT-SYS      ')
disp('=================================================================')
fprintf('\n\n'); disp('*======== semiarbitrary DLQR MATRICES for
the VIRTUAL DT-SYS  ==========*')
Qd = 1*eye(nx);  Rd=1*eye(nu); %---- WEIGHT MATRICES    PLAY WITH THEM !!!
disp('   ***  COMPUTE  DLQR for the VIRTUAL DT-SYS = Ar,Br,Cr   ***')%
[Kr, Pr, eigcl_r] = dlqr(Ar, Br, Qd, Rd);
%------ SHOW P(=RICCATI) & K (=LQR GAIN)
% disp('P=')   ;  disp(P); disp('Klqr=');  disp(Klqr)
disp('   *** Stability of the (virtual) DT closed-loop matrix
"Acl_r=Ar - Br*Kr" Guaranteed by DLQR    ***')

disp('=================================================================')
disp(' Now  Translate the "virtual" results back into the Original
CT system  ')
disp('=================================================================')
disp(' "Korig" =  Gain K  to be used on the  ORIGINAL  CT  SYS -> Korig = sqrt(r)*Kr'
Korig = sqrt(radius_ct)*Kr

%----- SS of CT CLOSED LOOP SYSTEM
Acl_orig_ctsys = A-B*Korig;
Bcl_orig_ctsys =B;               % if u=-Kx + r
Ccl_orig_ctsys =C-D*Korig ;      % if u=-Kx + r
Dcl=D; closed_orig_ctsys = ss(Acl_orig_ctsys,  Bcl_orig_ctsys,
Ccl_orig_ctsys,Dcl);

disp('eig(A-B*Korig) = '); disp(eig( Acl_orig_ctsys ))
```

```
disp('display eigcl: check whether abs( eigcl(i) - alpha_ct  ) <
radius_ct ') eigcl = eig( Acl_orig_ctsys ); fprintf('\n\n');
disp('YOUR  CHOICES  WERE...') alpha_ct radius_ct disp('   ***
Verify that ||eigcl - alpha_ct||  <  radius_ct  *** ...Press a
key...  ');  pause

for i=1:2
    disp('check that the norm: ||eigcl - alpha_ct|| <  radius_ct ')
    if abs( eigcl(i) - alpha_ct  ) <  radius_ct
        i
        disp('SUCCESS: eigcl_ct - alpha_ct < radius_ct ')
        norm( eigcl(i) - alpha_ct  )
        disp('   ***   Press any key   ***   '); pause
    else
        error('FAILURE  IN  D STAB !!!')
    end
end

disp('   pzmap...    ') %
fig=10; figure(fig);fig=fig+1; %
sgrid ; hold on ;  pzmap( closed_orig_ctsys ) ; hold off %
figure(fig);fig=fig+1; initial(open_orig_ctsys,closed_orig_ctsys,x0) % Compare  ZIR


fprintf('\n\n');
disp('==================================================================')
disp('            Disc-Stab for DISCRETE  TIME  SYS    ')
disp('==================================================================')

disp('==================================================================')
disp('  Discretize the orig CT-sys  with Ts= 1.0 ')
disp('==================================================================')
Ts= 1.0 ;     % Ts=1.333
%-------  CREATE DT-SS  ---------------%%
dtss_open = c2d(open_orig_ctsys, Ts, 'zoh') ;
%------- DT STATE SPACE DYNAMICS (Ad, Bd, Cd, Dd) ---------------%%
[Ad,Bd,Cd,Dd,Ts]=ssdata(dtss_open) ;

fprintf('\n\n');
disp('==================================================================')
disp(' Check Stability/Controllability of the Discretized DT-sys')
disp('==================================================================')
disp('DT open loop   eig/poles: eig(Ad)='); eig(Ad)%
if ldri_check_dt_eigs(Ad) ==1
    OPEN_LOOP_STAB_DT=1
    disp('***   Ad = PHI =OPEN-LOOP SYSTEM-MATRIX is SCHUR STABLE');
else
    OPEN_LOOP_STAB_DT=0
    disp('***   Ad = OPEN-LOOP SYSTEM-MATRIX is UNSTABLE !!!');
    %error('Ad = PHI =Delay-Free-SYSTEM-MATRIX is UNSTABLE ')
```

```
end
%-------
if (rank(ctrb(Ad,Bd)) ~= nx)
    CONTROLLABILITY_DT =0
    error('LDRI1: System (Ad,Bd)is NOT Controllable...Good Bye!!!!');
else
    CONTROLLABILITY_DT =1
    disp('LDRI1: System (Ad,Bd) is Controllable...Proceed');
end
%-----------------  D-STABILITY  Specifications -----------------%
disp('   Disc-STABILITY  = D(a,r)  WITHIN THE UNIT CIRCLE ') %
disp('alpha_dt = 0 = DESIRED Center  of  DISC for the closed DT sys     ') %
disp('radius_dt = 0.5 = DESIRED radius of   DISC --> must be  a
positive real number ')%
alpha_dt = 0.0 %
radius_dt = 0.5

disp('==================================================================')
disp('  VIRTUAL DT SYS = {Adr,Bdr,Cdr,Ddr} to be stabilized via DLQR or PLACE  ') %
disp('Adr = (1/radius_dt)*(Ad - alpha_dt*eye(nx)) is the
transformed Virtual OPEN-LOOP matrix')
disp('==================================================================')
Adr = (1/radius_dt)*(Ad - alpha_dt*eye(nx)); %
Bdr = Bd/sqrt(radius_dt) ; %
Cdr = Cd/sqrt(radius_dt) ; Ddr = Dd/radius_dt;

disp('==================================================================')
disp('  DLQR COMPUTATIONS  for the VIRTUAL DT-SYS{Adr,Bdr,Cdr,Ddr}
with the same semiarbitrary (Qd,Rd) matrices used for CT (this is
a legal option)  ')
disp('==================================================================')
fprintf('\n\n'); %
disp('*==  DLQR MATRICES for the VIRTUAL DT-SYS ==*') %
[Kdr, Pdr, eigcl_dr] = dlqr(Adr, Bdr, Qd, Rd); %
disp('   *** Stability of the (virtual) DT closed-loop matrix
"Acl_dr = Adr - Bdr*Kdr" Guaranteed by DLQR    ***') %
Acl_dr = Adr - Bdr*Kdr; %
disp('eig( Adr - Bdr*Kdr )'); disp(eig( Ar - Br*Kr ))

disp('==================================================================')
disp(' Now Translate the "virtual" results back into the Original
DT system  ')
disp('==================================================================')
disp(' "Korig_dt" =  Gain K  to be used on the  ORIGINAL  DT  SYS
--> Korig_dt = sqrt( radius_dt )*Kdr ')%
Korig_dt = sqrt(radius_dt)*Kdr

%----- SS of DT CLOSED LOOP SYSTEM
Acl_orig_dtsys = Ad-Bd*Korig_dt;
Bcl_orig_dtsys =Bd;               % if u=-Kx + r
```

```
Ccl_orig_dtsys =Cd-Dd*Korig_dt ;  % if u=-Kx + r
Dcl=Dd; %
closed_orig_dtsys = ss(Acl_orig_dtsys,  Bcl_orig_dtsys,
Ccl_orig_dtsys,Dcl, Ts); %
disp('eig(Ad-Bd*Korig_dt) = '); %
disp(eig( Acl_orig_dtsys )) %
eigcl_dt = eig( Acl_orig_dtsys );

disp('   *** Verify that ||eigcl_dt - alpha_dt||  <  radius_dt ');

for i=1:2
    disp('check that the norm: ||eigcl_dt - alpha_dt|| <  radius_dt ')
    if abs( eigcl_dt(i) - alpha_dt  ) <  radius_dt
        i
        disp('SUCCESS: eigcl - alpha < radius ')
        norm( eigcl_dt(i) - alpha_dt  )
        disp('   ***   Press any key   ***   '); pause
    else
        error('FAILURE  IN Disc STAB !!!')
    end
end

% pzmap of CLOSED LOOP SYSTEM
fig=10; figure(fig);fig=fig+1;%
zgrid ; hold on ; pzmap( closed_orig_dtsys ); hold off



%-----      END
```

14 Discrete Time Switched systems: stability analysis and control synthesis (INCOMPL)

The material in this subsection comes from the research work of Daafouz [28]. See also [29, 30, 31, 32, 33] for a tutorial introduction on switched systems. Reference [28] investigates the stability analysis and control synthesis of switched systems in the discrete domain. This class of "switched hybrid" systems is given by

$$x_{k+1} = A_\sigma x_k + B_\sigma u_k, \quad y_k = C_\sigma x_k \tag{213}$$

where the switching rule σ takes values in a finite set $I = \{1, ..., N\}$ which means that the matrices $A_\sigma, B_\sigma, C_\sigma$ are allowed to take values, at an arbitrary discrete time, in the finite set $(A_1, B_1, C_1), \ldots, (A_N, B_N, C_N)$.

The switching rule σ is NOT known a priori but we assume that its instantaneous value is available in real time (a rather realistic assumption where the switched system is supervised by a discrete–event system and the value of the discrete state is available in real time). The control synthesis is related to the design of a switched output feedback control $u_k = K_\sigma y_k$ ensuring stability of the closed–loop system

$$x_{k+1} = (A_\sigma + B_\sigma K_\sigma C_\sigma)x_k \tag{214}$$

The problem examined is the stability of the origin of an autonomous (discrete–time) switched system given by

$$x_{k+1} = A_\sigma x_k \tag{215}$$

Define the "indicator function" $s(k) = [s_1(k), ...s_N(k)]'$ with

$$s_i(k) = \begin{cases} 1, & \textit{when switched system is in } i_{th} \textit{ mode } (A_i) \quad , \\ 0, & \textit{otherwise i.e not in } i_{th} \textit{ mode } (\neq\ A_i) \quad . \end{cases} \tag{216}$$

The switched system $x_{k+1} = A_\sigma x_k$ can also be written as

$$x(k+1) = \sum_{i=1}^{N} s_i(k) A_i x(k) \tag{217}$$

Parameter–dependent Lyapunov functions have been used to check stability of polytopic time–varying systems. In the case of (217) this corresponds to the **switched Lyapunov function** defined as

$$V(k, x(k)) = x(k)^T P(s(k))x(k) = x(k)^T (\sum_{i=1}^{N} s_i(k) P_i) x(k) \tag{218}$$

with $P_1, ..., P_N$ symmetric positive–definite matrices. If such a positive–definite Lyapunov function exists and $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative–definite along the solutions of (217), then the origin of the switched system (215) is globally asymptotically stable as shown by the following general theorem from [**?**].

Theorem 90 *[**?**] The equilibrium "0" of $x(k+1) = f_k(x(k))$ is globally uniformly asymptotically stable if there is a function $V : Z^+ x R^n \to R$ such that (i) V is a positive–definite function, decrescent, and radially unbounded; (ii) $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative–definite along the solutions of $x(k+1) = f_k(x(k))$*

The Lyapunov function (218) is a positive–definite function, decrescent, and radially unbounded since

$$\begin{aligned} V(k,0) &= 0, \forall k \geq 0 \text{ and} \\ \beta_1 \|x_k\|^2 &\leq V(k,x(k)) = x(k)^T(\sum_{i=1}^{N} s_i(k)P_i)x(k) \leq \beta_2 \|x_k\|^2 \end{aligned} \tag{219}$$

for all $x(k) \in R^n, k \geq 0$ with $\beta_1 = min_{i\in I}\lambda_{min}(P_i)$ and $\beta_2 = max_{i\in I}\lambda_{max}(P_i)$ positive scalars.

The following theorem from [28] gives three equivalent **necessary and sufficient conditions** for the existence of a Lyapunov function of the form (218) whose difference is negative–definite, proving asymptotic stability of (215).

Theorem 91 *[28] The following statements are equivalent.*

- *There exists a Lyapunov of the form (218) whose difference is negative–definite, proving asymptotic stability of (215)*
- *There exist N symmetric matrices $P_1, ..., P_N$ satisfying*

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0 \ \ \forall (i,j) \in IxI \tag{220}$$

  The Lyapunov function is then given by $V(k,x(k)) = x(k)^T \left(\sum_{i=1}^{N} s_i(k)P_i\right) x(k)$

- *There exist N symmetric matrices $S_1, ..., S_N$ and N matrices $G_1, ..., G_N$ satisfying*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} > 0 \ \ \forall (i,j) \in IxI \tag{221}$$

  The Lyapunov function is then given by $V(k,x(k)) = x(k)^T \left(\sum_{i=1}^{N} s_i(k)S_i^{-1}\right) x(k)$.

Condition (220) has also been proposed in [34] to check stability of Piecewise Affine Systems.

15 Appendix A: Compendium of presented results (H_∞ & GCC)

15.1 H_∞ State Feedback Synthesis for CT LTI without Uncertainties

The CT–LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t), \ x(0) = 0 \\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \end{aligned}$$

is stabilizable via state feedback $u(t) = Kx(t)$ such that $\|T_{cl}(s)\|_\infty < \gamma$ **if and only if** there exist $S \in S^n$ (SPD matrix) and $Z \in \Re^{m \times n}$ such that

$$S > 0, \quad \begin{bmatrix} AS + B_u W + SA^T + W^T B_u^T & B_w & SC_z^T + W^T D_{zu}^T \\ B_w^T & -\gamma I & D_{zw}^T \\ C_z S + D_{zu} W & D_{zw} & -\gamma I \end{bmatrix} < 0 \tag{222}$$

If LMI (222) has a feasible solution (in terms of S, W, γ), the SSF control gain $K = WS^{-1}$ stabilizes the closed loop system robustly in the sense of "γ-attenuation".

15.2 H_∞ State Feedback Synthesis for CT LTI with Norm Bounded Uncertainties

Open–Loop System (Plant) with Norm Bounded Uncertainties:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + B_w w(t) + (B_u + \Delta B_u)u(t), \ x(0) = 0 \\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \end{aligned}$$

Norm Bounded Uncertainties:

$$[\Delta A \ \ \Delta B_u] = DF[E_a \ \ E_b], \quad F^T F \le I$$

State Feedback Controller:

$$u(t) = Kx(t)$$

Closed–Loop System:

$$\begin{aligned} \dot{x}(t) &= (A + B_u K + DF(E_a + E_b K))x(t) + B_w w(t) \triangleq A_{cl} x(t) + B_{cl} w(t) \\ z(t) &= (C_z + D_{zu} K)x(t) + D_{zw} w(t) \triangleq C_{cl} x(t) + D_{cl} w(t) \end{aligned}$$

Design Objective: Stabilization AND γ–attenuation (an H_∞ objective)

Solution:

$$\begin{bmatrix} (AS + B_u W) + (SA^T + W^T B_u^T) + \epsilon DD^T & B_w & SC_z^T + W^T D_{zu}^T & (E_a + E_b W)^T \\ * & -\gamma I & D_{zw}^T & 0 \\ * & * & -\gamma I & 0 \\ * & * & * & -\epsilon I \end{bmatrix} \tag{223}$$

If LMI (223) has a feasible solution (in terms of S, W, γ, ϵ), the SSF control gain $K = WS^{-1}$ stabilizes the closed loop system robustly in the sense of "γ-attenuation" for all admissible norm bounded uncertainties.

15.3 GCC problem setup

The Generic Case and Three Special (Sub)Cases of the GCC Approach

- The most Generic Case1: GCC Synthesis for uncertain DT system with state and input delays i.e. $x_{k+1} = (A + \Delta A)x_k + (A_1 + \Delta A_1)x_{k-d} + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$
- Case2: GCC Synthesis for uncertain DT systems with only Input Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h}$
- Case3: GCC Synthesis for uncertain DT systems with only State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k + (A_1 + \Delta A_1)x_{k-d}$
- Case4: GCC Synthesis for uncertain DT systems without Input or State Delay i.e. $x_{k+1} = (A + \Delta A)x_k + (B + \Delta B)u_k$

15.4 Result-1a: GCC SSF for Unc-DT-Sys with Input and State delay (most Generic Case)

The Generic Case Setup: Open–loop DT system with state and input delays and uncertain dynamics

$$x_{k+1} = (A + \Delta A)x_k + (A_1 + \Delta A_1)x_{k-d} + (B + \Delta B)u_k + (B_1 + \Delta B_1)u_{k-h} \tag{224}$$

with $x \in \Re^n$ and $u \in \Re^m$

- d and h are unknown constant integers representing the number of delay units in the state and input, respectively, bounded as $0 \le d \le d^*$, $0 \le h \le h^*$ with bounds d^*, h^* being known
- A, A_1, B, B_1 are known real constant matrices of appropriate dimensions
- uncertain matrices ΔA, ΔB, ΔA_1, ΔB_1 represent time-varying parameter uncertainties in the system model, satisfying

$$[\Delta A \;\; \Delta B \;\; \Delta A_1 \;\; \Delta B_1] = DF\,[E_a \;\; E_b \;\; E_d \;\; E_h] \tag{225}$$

- D, E_a E_b E_d E_h are known real constant matrices of appropriate dimensions describing the structure of uncertainties
- the unknown (time-varying) matrix F satisfies $F^T F \le I$, $\forall k$

Associated with the uncertain open–loop system (83) is the cost function

$$J = \sum_{k=0}^{\infty} \left[x_k^T Q x_k + u_k^T R u_k \right] \tag{226}$$

with $Q^T = Q > 0$, $R^T = R > 0$ being symmetric and positive definite (SPD) matrices of appropriate dimensions. Closing the loop in (83) with $u_k = Kx_k$, the closed–loop dynamics are

$$\begin{aligned} x_{k+1} &= [A + BK + DF(E_a + E_b K)]\,x_k + [B_1 + DFE_h]\,Kx_{k-h} + \\ &\quad [A_1 + DFE_d]\,x_{k-d} \\ &\triangleq A_C(k)x_k + B_H(k)Kx_{k-h} + A_D(k)x_{k-d} \end{aligned} \tag{227}$$

with the uncertain matrices $A_C,\ B_H,\ A_D$, defined as

$$A_C \triangleq A + BK + DF(E_a + E_b K),\quad B_H \triangleq B_1 + DFE_h,\quad A_D \triangleq A_1 + DFE_d \tag{228}$$

The cost function associated with the closed–loop system (86) is

$$J_{cl} = \sum_{k=0}^{\infty} x_k^T \left[Q + K^T R K\right] x_k \tag{229}$$

Sufficient condition for the existence of SSF GCC - (just a stepping stone - useless for computations)

Theorem 92 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller if there exist symmetric positive definite matrices $P,\ P_d \in \Re^{n\times n}, T \in \Re^{m\times m}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^T P A_D & A_C^T P B_H \\ * & A_D^T P A_D - P_d & A_D^T P B_H \\ * & * & B_H^T P B_H - T \end{bmatrix} < 0 \tag{230}$$

$$\Pi \ \triangleq\ A_C^T P A_C \underbrace{- P + P_d + K^T T K + Q + K^T R K} \triangleq A_C^T P A_C + \Lambda \tag{231}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices $A_C,\ B_H,\ A_D$ already defined in (87). Moreover the closed-loop cost function satisfies

$$\begin{aligned} J_{cl} \le J^* &\triangleq x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \\ &\le \lambda_{max}(U^T P U) + d^* \lambda_{max}(U^T P_d U) + h^* \lambda_{max}(U^T K^T T K U) \end{aligned} \tag{232}$$

Theorem 93 *For the uncertain system (83) and the cost function (85) there exist symmetric positive-definite matrices P, T such that matrix inequality (89) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}, M = P_d^{-1} \in \Re^{n\times n}, N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon D D^T & AS + BW & A_1 M & B_1 N & 0 & 0 & 0 & 0 & 0 \\ * & -S & 0 & 0 & (E_a S + E_b W)^T & S^T & W^T & S & W^T \\ * & * & -M & 0 & M E_d^T & 0 & 0 & 0 & 0 \\ * & * & * & -N & N E_h^T & 0 & 0 & 0 & 0 \\ * & * & * & * & -\epsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -M & 0 & 0 & 0 \\ * & * & * & * & * & 0 & -N & 0 & 0 \\ * & * & * & * & * & 0 & 0 & -Q^{-1} & 0 \\ * & * & * & * & * & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0. \tag{233}$$

Furthermore, if matrix inequality (189) has a feasible solution in terms of the variables {ϵ, W, S, M, N } then the state feedback control law $u_k = WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies

$$J \le (1 + h^*)\lambda_{\max}(U^T S^{-1} U) + d^* \lambda_{\max}(U^T M^{-1} U) \tag{234}$$

15.5 Result-1b: SSF Synthesis for Unc-DT-Sys with Input and State delay - no GCC, only Robust Stabilization

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 69, it is easy to prove the following Corollary.

Corollary 94 *For the uncertain system (83) (with input and state delays) there exist symmetric positive-definite matrices P, P_d, T such that $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ (with $V_k^{tot} = x_k^T P x_k + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j} + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}$ defined in (191)) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrix $S = P^{-1} \in \Re^{n\times n}$, $M = P_d^{-1} \in \Re^{m\times m}$, $N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & A_1M & B_1N & 0 & 0 & 0 \\ * & -S & 0 & 0 & (E_aS+E_bW)^T & S & W^T \\ * & * & -M & 0 & ME_d^T & 0 & 0 \\ * & * & * & -N & NE_h^T & 0 & 0 \\ * & * & * & * & -\epsilon I & 0 & 0 \\ * & * & * & * & * & -M & 0 \\ * & * & * & * & * & 0 & -N \end{bmatrix} < 0. \tag{235}$$

Furthermore, if matrix inequality (190) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$, then the state feedback control law $u_k = WS^{-1}x_k = Kx_k$ is a robustly stabilizing control law. □

LMIs (190) is a "subset of the Generic" LMI (189) formally derived after removing the "appropriate" rows and columns i.e. the last two rows and columns containing the matrices Q, R.

15.6 Result-2a: GCC SSF for Unc-DT-Sys with Input Delay (only)

Open–loop DT system with state and input delay (NO STATE DELAY $\Rightarrow A_1 = \Delta A_1 = E_d = 0$) and uncertain dynamics

$$x_{k+1} = (A+\Delta A)x_k + (B+\Delta B)u_k + (B_1+\Delta B_1)u_{k-h} \tag{236}$$

with $x \in \Re^n$ and $u \in \Re^m$ and (since $A_1 = \Delta A_1 = E_d = 0$)

$$[\Delta A \; \Delta B \; \Delta B_1] = DF\,[E_a \; E_b \; E_h] \tag{237}$$

with unknown (time-varying) matrix F satisfying $F^TF \leq I$. Furthermore h is an unknown constant integer (delay units in the input), bounded as $0 \leq h \leq h^*$ with h^* known.

Same cost function (85) as before.

The closed–loop dynamics with $u_k = Kx_k$ are

$$\begin{aligned} x_{k+1} &= [A+BK+\Delta A+\Delta BK]\,x_k + [B_1+\Delta B_1]\,Kx_{k-h} \\ &= [A+BK+DF(E_a+E_bK)]\,x_k + [B_1+DFE_h]\,Kx_{k-h} \\ &\triangleq A_C(k)x_k + B_H(k)Kx_{k-h} \end{aligned} \tag{238}$$

with the uncertain matrices A_C, B_H, defined as

$$A_C \triangleq A+BK+DF(E_a+E_bK), \quad B_H \triangleq B_1+DFE_h \tag{239}$$

Sufficient condition for the existence of SSF GCC (just a stepping stone - useless for computations): The sufficient condition for the existence of memoryless state feedback GCC law is a special "case" of Theorem (68)

Theorem 95 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller for (110) if there exist symmetric positive definite matrices $P \in \Re^{n\times n}, T \in \Re^{m\times m}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^T P B_H \\ * & B_H^T P B_H - T \end{bmatrix} < 0 \tag{240}$$

$$\Pi \triangleq A_C^T P A_C \underbrace{-P + K^T T K + Q + K^T R K}{} \triangleq A_C^T P A_C + \Lambda \tag{241}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices A_C, B_H already defined in (113). Moreover the closed-loop cost function satisfies

$$J_{cl} \leq J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{h} x_{-i}^T K^T T K x_{-i} \leq \lambda_{max}(U^T P U) + h^* \lambda_{max}(U^T K^T T K U) \tag{242}$$

GCC Synthesis for systems with (only) INPUT DELAY

Theorem 96 *For the uncertain (input delayed) system (110) and the cost function (85) there exist symmetric positive-definite matrices P, T such that matrix inequality (114) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon D D^T & AS + BW & B_1 N & 0 & 0 & 0 & 0 \\ * & -S & 0 & (E_a S + E_b W)^T & W^T & S & W^T \\ * & * & -N & N E_h^T & 0 & 0 & 0 \\ * & * & * & -\epsilon I & 0 & 0 & 0 \\ * & * & * & * & -N & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & * & -R^{-1} \end{bmatrix} < 0. \tag{243}$$

Furthermore, if matrix inequality (123) has a feasible solution in terms of the variables $\{\epsilon, W, S, N\}$ then the state feedback control law $u_k = W S^{-1} x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies $J \leq (1 + h^*)\lambda_{\max}(U S^{-1} U)$.

Note that the LMI (123) of Theorem 46, results from the "Generic" LMI (189) after removing the third row/column (containing M, A_1, E_d which are "zero" matrices since they involve state-delay) and the sixth row/column (involve M, S) with M being a "zero" while matrix S is already constrained via LMI (123)

15.7 Result-2b: SSF Synthesis for Unc-DT-Sys with Input Delay (only) - no GCC, only Robust Stabilization

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 46, it is easy to prove the following Corollary.

Corollary 97 *For the uncertain (input delayed) system (110) there exist symmetric positive-definite matrices P, T such that $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ (with V_k^{tot} defined as $V_k^{tot} = x_k^T P x_k + \sum_{j=1}^{h} x_{k-j}^T K^T T K x_{k-j}$) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix*

$W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $N = T^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.

$$\begin{bmatrix} -S+\epsilon DD^T & AS+BW & B_1N & 0 & 0 \\ * & -S & 0 & (E_aS+E_bW)^T & W^T \\ * & * & -N & NE_h^T & 0 \\ * & * & * & -\epsilon I & 0 \\ * & * & * & * & 0 \\ * & * & * & * & -N \end{bmatrix} < 0. \tag{244}$$

Furthermore, if matrix inequality (131) has a feasible solution, in terms of the variables $\{\epsilon, W, S\}$, then the state feedback control law $u_k = WS^{-1}x_k = Kx_k$ is a robustly stabilizing control law. □

LMIs (131) is a "subset of the Generic" LMI (123) formally derived after removing the "appropriate" rows and columns i.e. the last two rows and columns containing the matrices Q, R.

15.8 Result-3a: GCC SSF for Unc-DT-Sys with State Delay (only)

Same cost function (85) as before.

Open–loop DT system with (only) state delay and uncertain dynamics

$$x_{k+1} = (A+\Delta A)x_k + (A_1+\Delta A_1)x_{k-d} + (B+\Delta B)u_k \tag{245}$$

with $x \in \Re^n$ and $u \in \Re^m$ and NO INPUT DELAY (hence $B_1 = \Delta B_1 = E_h = 0$)

$$[\Delta A\ \Delta A_1\ \Delta B] = DF\,[E_a\ E_d\ E_b] \tag{246}$$

with the unknown (time-varying) matrix F satisfying $F^TF \le I$. Furthermore d is an unknown constant integer (delay units in the state), bounded as $0 \le d \le d^*$ with d^* known. The closed–loop dynamics with $u_k = Kx_k$ are

$$\begin{aligned} x_{k+1} &= [A+BK+DF(E_a+E_bK)]\,x_k + [A_1+DFE_d]\,x_{k-d} \\ &\triangleq A_C(k)x_k + A_D(k)x_{k-h} \end{aligned} \tag{247}$$

with the uncertain matrices A_C, A_D, defined as

$$A_C \triangleq A+BK+DF(E_a+E_bK), \quad A_D \triangleq A_1+DFE_d \tag{248}$$

Sufficient condition for the existence of SSF GCC for Systems with (only) STATE DELAY (just a stepping stone - useless for computations): The sufficient condition for the existence of memoryless state feedback GCC law is a special "case" of Theorem (68)

Theorem 98 *The control law $u_k^* = Kx_k$ is a guaranteed cost controller for (132) if there exist symmetric positive definite matrices P, $P_d \in \Re^{n\times n}$ such that for any admissible uncertain matrix F the following matrix inequality holds:*

$$\begin{bmatrix} \Pi & A_C^TPA_D \\ * & A_D^TPA_D - P_d \end{bmatrix} < 0 \tag{249}$$

$$\Pi \triangleq A_C^TPA_C \underbrace{-P+P_d+Q+K^TRK}_{} \triangleq A_C^TPA_C + \Lambda \tag{250}$$

with the obvious definition for Λ and the uncertain closed–loop system matrices A_C, A_D already defined in (135). Moreover the closed-loop cost function satisfies

$$J_{cl} \leq J^* \triangleq x_0^T P x_0 + \sum_{i=1}^{d} x_{-i}^T P_d x_{-i} \leq \lambda_{max}(U^T P U) + d^* \lambda_{max}(U^T P_d U) \quad (251)$$

GCC Synthesis for systems with (only) STATE DELAY

Theorem 99 *For the uncertain (state delayed) system (132) and the cost function (85) there exist symmetric positive-definite matrices P, P_d such that matrix inequality (136) holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $M = P_d^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & A_1 M & 0 & 0 & 0 & 0 \\ (AS + BW)^T & -S & 0 & (E_a S + E_b W)^T & S & S & W^T \\ MA_1^T & 0 & -M & ME_d^T & 0 & 0 & 0 \\ 0 & (E_a S + E_b W) & E_d M & -\epsilon I & 0 & 0 & 0 \\ 0 & S & 0 & 0 & -M & 0 & 0 \\ 0 & S & 0 & 0 & 0 & -Q^{-1} & 0 \\ 0 & W & 0 & 0 & 0 & 0 & -R^{-1} \end{bmatrix} < 0. \quad (252)$$

Furthermore, if matrix inequality (145) has a feasible solution in terms of the variables $\{\epsilon, W, S\}$ then the state feedback control law $u_k = WS^{-1}x_k$ is a guaranteed cost control law and the corresponding closed-loop cost function satisfies $J \leq (d^*)\lambda_{\max}(US^{-1}U)$

Note that the LMI (145) of Theorem 53, results from the "Generic" LMI (189) after removing the fourth row/column (involve B_1, N) and the seventh row/column (involve W, N).

15.9 Result-3b: SSF Synthesis for Unc-DT-Sys with State Delay (only) - no GCC, only Robust Stabilization

If the demand for guaranteed cost is alleviated, following the same lines of the proof of Theorem 53, it is easy to prove the following Corollary.

Corollary 100 *For the uncertain (state delayed) system (132) there exist symmetric positive-definite matrices P, T such that $\Delta V_k = V_{k+1}^{tot} - V_k^{tot} < 0$ (with $V_k^{tot} = x_k^T P x_k + \sum_{i=1}^{d} x_{k-i}^T P_d x_{k-i}$ holds for all admissible uncertainties if and only if there exist a positive scalar $\epsilon > 0$, a matrix $W \in \Re^{m\times n}$ and symmetric positive definite matrices $S = P^{-1} \in \Re^{n\times n}$, $M = P_d^{-1} \in \Re^{m\times m}$ such that the following LMI is satisfied.*

$$\begin{bmatrix} -S + \epsilon DD^T & AS + BW & A_1 M & 0 & 0 \\ (AS + BW)^T & -S & 0 & (E_a S + E_b W)^T & S \\ MA_1^T & 0 & -M & ME_d^T & 0 \\ 0 & (E_a S + E_b W) & E_d M & -\epsilon I & 0 \\ 0 & S & 0 & 0 & -M \\ 0 & S & 0 & 0 & 0 \\ 0 & W & 0 & 0 & 0 \end{bmatrix} < 0. \quad (253)$$

Furthermore, if matrix inequality (154) has a feasible solution in terms of the variables $\{\epsilon, W, S\}$ then the state feedback control law $u_k = WS^{-1}x_k = Kx_k$ is a robustly stabilizing control law.

16 Appendix B: The three "benchmark" systems used in simulations

The following SISO systems with $x \in \Re^n, \;\; n = 2$ and $u \in \Re^m, \;\; m = 1$ are used as "benchmark" systems in the simulation sections.

16.1 The Stable Minimum Phase "benchmark" system

State Space Equations

The continuous–time system with transfer function $G(s) = \frac{2}{s^2+3s+2}$, is a SISO open–loop stable system with the following state space description.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t), \;\; y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \,. \tag{254}$$

16.2 The Stable Nonmiminimum phase "benchmark" system

State Space Equations

The non–minimum phase open–loop stable continuous–time system $G_9(s) = \frac{-6s+3}{50s^2+15s+1}$ from [5], with a state space description,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -0.02 & -0.30 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \;\; y(t) = \begin{bmatrix} 0.0600 & -0.1200 \end{bmatrix} x(t) \tag{255}$$

16.3 The Unstable "benchmark" system

State Space Equations

The continuous–time system with transfer function $G(s) = \frac{0.1}{s^2+0.1s}$ is a SISO open–loop unstable system extensively used as "benchmark" system in NCS literature [35, 36, 37, 38]. Its state–space description is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \;\; y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{256}$$

2-BE CONTINUED... 12Nov2011, 12Nov2020

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